Appl. Set-Valued Anal. Optim. 6 (2024), No. 1, pp. 103-112
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.6.2024.1.09

ON AN INERTIAL KRASNOSEL'SKIĬ-MANN ITERATION

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Dedicated to Professor Simeon Reich on the Occasion of His 75th Birthday

Abstract. In this paper, we consider the Krasnosel'skiĭ-Mann iteration for finding a fixed point of non-expansive operators in real Hilbert spaces. By introducing new, concise, and self-contained techniques, we propose new inertial factors with desirable upper bounds better than or comparable to existing ones.
Keywords. Inertial factors; Krasnosel'skiĭ-Mann iteration; Non-expansive operator; Weak convergence.

1. INTRODUCTION

The fixed point problem of non-expansive operators in real Hilbert spaces is a fundamental problem in mathematics. The classical iterative scheme is the famous Banach iteration. It requires the associate operator to be a strictly contractive mapping. To circumvent this, one may resort to the Krasnosel'skiĭ-Mann (KM for short) iteration [1, 2, 3] whose next iterate is a convex combination of the current iterate and its operator evaluation

$$x_{k+1} = (1-\alpha_k)x_k + \alpha_k T(x_k), \quad k = 0,1,...,$$

where the coefficient α_k is in $[0,1]$, the series $\sum \alpha_k(1-\alpha_k)$ diverges. For recent pertinent discussions, we refer to [4, 5, 6, 7, 8, 9] and the references cited therein.

In particular, inspired by a pioneering work on the proximal point algorithm [10], Maingé [4] suggested adding an inertial term to the KM iteration in the following way

$$\begin{aligned} y_k &= x_k + t_k(x_k - x_{k-1}), \\ x_{k+1} &= (1-\alpha_k)y_k + \alpha_k T(y_k), \quad k = 0,1,..., \end{aligned} \tag{1.1}$$

where $t_k \geq 0$ is termed an inertial factor. To prove the weak convergence of the inertial KM iteration described by (1.1), Maingé [4] gave the following conditions on inertial factors, which depend on the iterates:

$$0 \leq t_k \leq t < 1, \quad t_k\|x_k - x_{k-1}\|^2 \text{ is summable.} \tag{1.2}$$

In practical implementations, one may take

$$t_k\|x_k - x_{k-1}\|^2 \leq c/(k+1)^2, \tag{1.3}$$

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Received 22 October 2023; Accepted 29 November 22023; Published online 21 December 2023

where $c \geq 0$ is the control factor. From a different viewpoint, Boţ, Csetnek, and Hendrich [5] proposed new conditions, which are free of iterates; see (2.1) below. In our opinion, they are more desirable than (1.2) as how to choose the control factor c in (1.3) remains open. Recently, Dong [6] improved these results (see (2.2) and (2.3) below), and Fierro, Maulén, and Peypouquet [9] obtained better upper bounds of inertial factors by refining convergence analysis in [6]; see (2.4) below. Notice that the seminal lemma [10, Lemma 2.3] was involved in [4, 5, 6] explicitly and [9, Theorem 4] implicitly.

In this paper, we aim at taking advantage of some newly-developed mathematical techniques [11] to introduce different assumptions regarding the inertial factors of the KM iteration. Furthermore, the resulting upper bounds are desirably larger than those [6] and comparable to the latest results in [9, Theorem 4]. Unlike [4, 5, 6, 9] , our proof no longer relies on the above-mentioned lemma and is new, concise, and self-contained.

2. Preliminaries

In this section, we first give some basic definitions and then provide some auxiliary results for later use.

Let $\mathscr{H}$ be a real Hilbert space, in which $\langle x,y\rangle$ stands for the usual inner product and $\|x\| := \sqrt{\langle x,x\rangle}$ for the induced norm for any $x,y \in \mathscr{H}$.

Recall that an operator $T : \mathscr{H} \to \mathscr{H}$ is called non-expansive if and only if

$$\|T(x) - T(y)\| \leq \|x - y\|, \quad \forall\, x,\, y \in \mathscr{H}.$$

Now we review the conditions on the inertial factors [5, 6, 9] in details. Specially speaking, the conditions in [5] are that $\{t_k\}$ and $\{\alpha_k\}$ satisfy

$$\begin{aligned} &0 \leq t_k \leq t_{k+1} \leq t < 1, \\ &0 < \liminf \alpha_k \leq \alpha_k \leq \frac{\delta(1-t^2) - t^2 - t^3 - t\sigma}{\delta[1 + t(1+t) + t\delta + \sigma]}, \quad \text{with } \delta > \frac{t^2 + t^3 + t\sigma}{1 - t^2}, \end{aligned} \tag{2.1}$$

where $\delta > 0$ and $\sigma > 0$. The conditions in [6] are

$$0 \leq t_k \leq t_{k+1} \leq 1, \;\; \varepsilon \leq \alpha_k \leq 1, \tag{2.2}$$

where ε is any given sufficiently small positive number. Particularly, in the $\alpha_k \in (0.5, 1-\varepsilon]$ case, it follows from [6, Theorem 3.1] that

$$t_{k+1} \leq t(\alpha_k, \alpha_{k+1}, \varepsilon) := \sqrt{p_k^2 + q_k} - p_k, \tag{2.3}$$

where p_k and q_k are given by

$$p_k := \frac{1}{2}\frac{1}{2 - \frac{1}{\alpha_{k+1}}}\left(\frac{1}{\alpha_k} + \frac{1}{\alpha_{k+1}} - 1\right), \quad q_k := \frac{1}{2 - \frac{1}{\alpha_{k+1}}}\left(\frac{1}{\alpha_k} - 1 - \varepsilon\right).$$

Finally, we give the conditions in [9]

$$\begin{aligned} &0 \leq t_k \leq t_{k+1} < 1, \\ &t_k(1+t_k) + (\alpha_k^{-1} - 1)t_k(1-t_k) \leq (\alpha_{k-1}^{-1} - 1)(1 - t_{k-1}) - \varepsilon, \end{aligned} \tag{2.4}$$

where $\varepsilon \geq 0$. Obviously, they are distinct from (2.2)-(2.3) as t_k can be bounded above by not only α_{k-1}, α_k, and $\varepsilon \geq 0$, but also t_{k-1}.

3. Main Results

In this section, we study an inertial KM iteration. Under weaker assumptions on inertial factors, we analyze its weak convergence.

First of all, we would like to point out that, by our numerical experiments in [6], α_k in the KM iteration is close to 1 for numerical efficiency in practice and weak convergence in theory. Thus, we give a practical, inertial KM iteration —Algorithm 1.

Algorithm 1 A practical inertial KM iteration

Step 0 Choose $x_{-1} = x_0 \in \mathscr{H}$. Choose $\varepsilon = 10^{-9}$ and $\alpha \in [0.80, 0.99]$. Compute $h_2(\alpha)$ via (3.1) and denote by $\bar{t}^+$. Choose $t_{-1} = 0$. Set $k := 0$.

Step 1 Choose $t_k \in [t_{k-1}, \bar{t}^+]$. Compute

$$y_k = x_k + t_k(x_k - x_{k-1}), \quad x_{k+1} = (1-\alpha)y_k + \alpha T(y_k).$$

Set $k := k+1$.

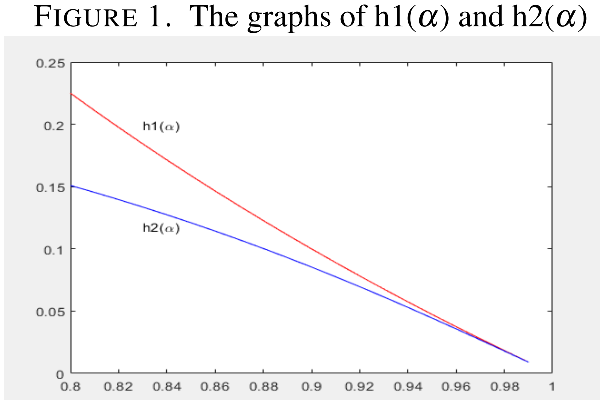
To provide a better understanding of the practical inertial KM iteration, we define $h_1(\alpha)$ and $h_2(\alpha)$ as follows

$$\begin{aligned} h_1(\alpha) &:= 1 - \varepsilon - \left(1 + 0.9(\alpha^{-1} - 1)\right)^{-1} = 1 - \varepsilon - (0.1 + 0.9\alpha^{-1})^{-1}, \\ h_2(\alpha) &:= 0.5 \frac{-1 + \sqrt{1 + 4(9\alpha^{-1} - 8)\,(0.9(\alpha^{-1} - 1) - \varepsilon)}}{9\alpha^{-1} - 8} \ (:= \bar{t}^+), \end{aligned} \tag{3.1}$$

where $\varepsilon = 10^{-9}$, and the graphs of $h_1(\alpha)$ and $h_2(\alpha)$ are plotted in Figure 1. Furthermore, in the $\varepsilon = 0$ case, it is direct to check that

$$h_1(\alpha) > h_2(\alpha), \quad \alpha \in [0.80, 0.99].$$

FIGURE 1. The graphs of h1(α) and h2(α)



Below, we consider the inertial KM iteration described by (1.1) in a general case of $\alpha_k \in [\varepsilon, 1-\varepsilon]$, for a given sufficiently small positive number ε. We assume that (i) the sequences $\{\alpha_k\}$ and $\{t_k\}$ satisfy

$$\alpha_k \in [\varepsilon, 1-\varepsilon], \quad 1 \geq t_k \geq t_{k-1} \geq 0; \tag{3.2}$$

(ii)

$$t_k^+ := 0.5\frac{-1+\sqrt{1+4\left((\sigma^{-1}-1)(\alpha_k^{-1}-1)+1\right)\left((1-\sigma)(\alpha_{k-1}^{-1}-1)-\varepsilon\right)}}{(\sigma^{-1}-1)(\alpha_k^{-1}-1)+1},$$

$$t_k \le \min\left\{t_k^+, 1-\varepsilon-\left(1+(1-\sigma)(\alpha_{k-1}^{-1}-1)\right)^{-1}\right\}, \tag{3.3}$$

where σ is properly chosen in $(0,1)$ in advance.

Obviously, for this inertial KM iteration, where $\{\alpha_k\}$ and $\{t_k\}$ satisfy assumptions (3.2)-(3.3), it reduces to Algorithm 1 provided that $\alpha_k \equiv \alpha$ and $\sigma = 0.1$.

Notice that we shall maximize the corresponding upper bound in (3.3), which is the function with respect to $\sigma \in (0,1)$; see Remark 3.2 and Figure 2 below for more details.

To prove the weak convergence for the inertial KM iteration (1.1) under above assumptions, we introduce the following two lemmas, used in [11], to simplify the analysis.

Lemma 3.1. *Let $\alpha > 0$. If $4\alpha\beta \ge \gamma^2$, then $\alpha\|a\|^2 + \beta\|b\|^2 + \gamma\langle a,b\rangle \ge 0$ for all a, $b \in \mathscr{H}$.*

Proof. For any given a, $b \in \mathscr{H}$, the following inequality holds

$$\alpha\|a\|^2 + \beta\|b\|^2 + \gamma\langle a,b\rangle \ge (2\sqrt{\alpha\beta} - |\gamma|)\|a\|\|b\|.$$

Since $4\alpha\beta \ge \gamma^2$, we have

$$\alpha\|a\|^2 + \beta\|b\|^2 + \gamma\langle a,b\rangle \ge 0.$$

The proof is complete. □

Lemma 3.2. [12] *Let $\alpha > 0$ and $t \in R$. If $4\alpha > t^2\beta$, then, for all $x, u \in \mathscr{H}$,*

$$\langle x, \alpha x\rangle + \langle u, \beta u\rangle - t\langle x, \beta u\rangle \ge \frac{\alpha+\beta-\sqrt{(\alpha-\beta)^2+t^2\beta^2}}{2}(\|x\|^2+\|u\|^2).$$

Using these lemmas, we can establish the weak convergence of the inertial KM iteration.

Theorem 3.1. *If the assumptions (3.2)-(3.3) hold, then the sequence $\{x_k\}$ generated by the inertial KM iteration (1.1) in the general case of $\alpha_k \in [\varepsilon, 1-\varepsilon]$, converges weakly to a fixed point of T.*

Proof. For any given fixed point z of T, i.e., $T(z) = z$, we obtain

$$\begin{aligned}
&\|x_{k+1}-z\|^2\\
&= (1-\alpha_k)^2\|y_k-z\|^2 + \alpha_k^2\|T(y_k)-T(z)\|^2 + 2(1-\alpha_k)\alpha_k\langle y_k-z, T(y_k)-T(z)\rangle\\
&= (1-\alpha_k)^2\|y_k-z\|^2 + \alpha_k^2\|T(y_k)-T(z)\|^2\\
&\quad + (1-\alpha_k)\alpha_k(\|T(y_k)-T(z)\|^2 + \|y_k-z\|^2 - \|T(y_k)-y_k\|^2)\\
&= (1-\alpha_k)\|y_k-z\|^2 + \alpha_k\|T(y_k)-T(z)\|^2 - \alpha_k(1-\alpha_k)\|T(y_k)-y_k\|^2\\
&\le \alpha_k\|y_k-z\|^2 + (1-\alpha_k)\|y_k-z\|^2 - \alpha_k(1-\alpha_k)\|T(y_k)-y_k\|^2.
\end{aligned}$$

From (1.1), we have

$$\alpha_k(T(y_k)-y_k) = x_{k+1}-y_k = x_{k+1}-x_k-t_k(x_k-x_{k-1}),$$

so

$$\alpha_k^2\|T(y_k)-y_k\|^2 = \|x_{k+1}-x_k\|^2 + t_k^2\|x_k-x_{k-1}\|^2 - 2t_k\langle x_{k+1}-x_k, x_k-x_{k-1}\rangle.$$

Observe that

$$\begin{aligned}
&\|y_k - z\|^2 \\
&= (1+t_k)^2\|x_k - z\|^2 + t_k^2\|x_{k-1} - z\|^2 - 2(1+t_k)t_k\langle x_k - z, x_{k-1} - z\rangle \\
&= (1+t_k)^2\|x_k - z\|^2 + t_k^2\|x_{k-1} - z\|^2 - (1+t_k)t_k(\|x_k - z\|^2 + \|x_{k-1} - z\|^2 - \|x_k - x_{k-1}\|^2) \\
&= (1+t_k)\|x_k - z\|^2 - t_k\|x_{k-1} - z\|^2 + t_k(1+t_k)\|x_k - x_{k-1}\|^2.
\end{aligned}$$

Thus, we further have

$$\begin{aligned}
&\|x_{k+1} - z\|^2 \\
&\leq (1+t_k)\|x_k - z\|^2 - t_k\|x_{k-1} - z\|^2 - (\alpha_k^{-1} - 1)\|x_{k+1} - x_k\|^2 \\
&\quad + 2(\alpha_k^{-1} - 1)t_k\langle x_{k+1} - x_k, x_k - x_{k-1}\rangle + \big(t_k(1+t_k) - (\alpha_k^{-1} - 1)t_k^2\big)\|x_k - x_{k-1}\|^2.
\end{aligned}$$

From assumption (3.2), we have

$$\begin{aligned}
&\|x_{k+1} - z\|^2 - t_{k+1}\|x_k - z\|^2 + (1-\sigma)(\alpha_k^{-1} - 1)\|x_{k+1} - x_k\|^2 \\
&\leq \|x_k - z\|^2 - t_k\|x_{k-1} - z\|^2 + (1-\sigma)(\alpha_{k-1}^{-1} - 1)\|x_k - x_{k-1}\|^2 - \Delta_k,
\end{aligned}$$

where $\sigma \in (0,1)$ and Δ_k is given by

$$\begin{aligned}
\Delta_k := {}& \sigma(\alpha_k^{-1} - 1)\|x_{k+1} - x_k\|^2 - 2(\alpha_k^{-1} - 1)t_k\langle x_{k+1} - x_k, x_k - x_{k-1}\rangle \\
&+ \big((1-\sigma)(\alpha_{k-1}^{-1} - 1) - t_k(1+t_k) + (\alpha_k^{-1} - 1)t_k^2\big)\|x_k - x_{k-1}\|^2. \qquad (3.4)
\end{aligned}$$

Set

$$\varphi_k := \|x_k - z\|^2 - t_k\|x_{k-1} - z\|^2 + (1-\sigma)(\alpha_{k-1}^{-1} - 1)\|x_k - x_{k-1}\|^2.$$

Then

$$\varphi_{k+1} \leq \varphi_k - \Delta_k. \qquad (3.5)$$

Next, we choose t_k to guarantee that

$$\varphi_k \geq \varepsilon\|x_{k-1} - z\|^2, \quad \Delta_k \geq \varepsilon\|x_k - x_{k-1}\|^2.$$

Consider

$$\begin{aligned}
\varphi_k &:= \|x_k - z\|^2 - t_k\|x_{k-1} - z\|^2 + (1-\sigma)(\alpha_{k-1}^{-1} - 1)\|x_k - x_{k-1}\|^2 \\
&= \|x_{k-1} - z\|^2 + 2\langle x_{k-1} - z, x_k - x_{k-1}\rangle + \|x_k - x_{k-1}\|^2 \\
&\quad - t_k\|x_{k-1} - z\|^2 + (1-\sigma)(\alpha_{k-1}^{-1} - 1)\|x_k - x_{k-1}\|^2 \\
&= (1-t_k)\|x_{k-1} - z\|^2 + 2\langle x_{k-1} - z, x_k - x_{k-1}\rangle + \big(1 + (1-\sigma)(\alpha_{k-1}^{-1} - 1)\big)\|x_k - x_{k-1}\|^2 \\
&= \varepsilon\|x_{k-1} - z\|^2 + (1 - t_k - \varepsilon)\|x_{k-1} - z\|^2 + 2\langle x_{k-1} - z, x_k - x_{k-1}\rangle \\
&\quad + \big(1 + (1-\sigma)(\alpha_{k-1}^{-1} - 1)\big)\|x_k - x_{k-1}\|^2.
\end{aligned}$$

Combining this with Lemma 3.1 and the assumption (3.3)

$$\begin{aligned}
&4\,(1 - \varepsilon - t_k)\big(1 + (1-\sigma)(\alpha_{k-1}^{-1} - 1)\big) \geq 2^2 \\
\Leftrightarrow\quad & t_k \leq 1 - \varepsilon - \big(1 + (1-\sigma)(\alpha_{k-1}^{-1} - 1)\big)^{-1}
\end{aligned}$$

yields $\varphi_k \geq \varepsilon\|x_{k-1}-z\|^2$. Similarly, it follows from (3.4)

$$\begin{aligned}\Delta_k := & \sigma(\alpha_k^{-1}-1)\|x_{k+1}-x_k\|^2 - 2(\alpha_k^{-1}-1)t_k\langle x_{k+1}-x_k, x_k-x_{k-1}\rangle \\ & + \left((1-\sigma)(\alpha_{k-1}^{-1}-1) - t_k(1+t_k) + (\alpha_k^{-1}-1)t_k^2\right)\|x_k-x_{k-1}\|^2 \\ = & \sigma(\alpha_k^{-1}-1)\|x_{k+1}-x_k\|^2 - 2(\alpha_k^{-1}-1)t_k\langle x_{k+1}-x_k, x_k-x_{k-1}\rangle + \varepsilon\|x_k-x_{k-1}\|^2 \\ & + \left((1-\sigma)(\alpha_{k-1}^{-1}-1) - t_k(1+t_k) + (\alpha_k^{-1}-1)t_k^2 - \varepsilon\right)\|x_k-x_{k-1}\|^2.\end{aligned}$$

By Lemma 3.1 and assumption (3.3)

$$\begin{aligned} & 4\,\sigma(\alpha_k^{-1}-1)\left((1-\sigma)(\alpha_{k-1}^{-1}-1) - t_k(1+t_k) + (\alpha_k^{-1}-1)t_k^2 - \varepsilon\right) \geq 2^2\,(\alpha_k^{-1}-1)^2 t_k^2 \\ \Leftrightarrow\quad & \left((\sigma^{-1}-1)(\alpha_k^{-1}-1)+1\right)t_k^2 + t_k - (1-\sigma)(\alpha_{k-1}^{-1}-1) + \varepsilon \leq 0 \\ \Leftrightarrow\quad & t_k \leq 0.5\frac{-1+\sqrt{1+4\left((\sigma^{-1}-1)(\alpha_k^{-1}-1)+1\right)\left((1-\sigma)(\alpha_{k-1}^{-1}-1)-\varepsilon\right)}}{(\sigma^{-1}-1)(\alpha_k^{-1}-1)+1},\end{aligned}$$

we can derive $\Delta_k \geq \varepsilon\|x_k-x_{k-1}\|^2$. Obviously, from the two relations and (3.5), we conclude that

$$\begin{aligned}\lim \varphi_k \text{ exists} \quad &\Rightarrow \quad \|x_{k-1}-z\| \ (\text{thus } \|x_k-z\|) \text{ is bounded in norm;} \\ \lim \Delta_k = 0 \quad &\Rightarrow \quad \lim \|x_k-x_{k-1}\| = 0.\end{aligned}$$

These facts indicate that $\{x_k\}$ is bounded in norm. Thus it has at least one weak cluster point, say x^∞, i.e., there exists some subsequence $\{x_{k_j}\}$ converging weakly to x^∞. Meanwhile, we have

$$(I-T)\left(x_k + t_k(x_k-x_{k-1})\right) = \frac{t_k(x_k-x_{k-1}) - (x_{k+1}-x_k)}{\alpha_k},$$

so

$$\left\|\frac{t_k(x_k-x_{k-1}) - (x_{k+1}-x_k)}{\alpha_k}\right\| \leq \frac{t_k\|x_k-x_{k-1}\| + \|x_{k+1}-x_k\|}{\alpha_k}.$$

Note that $\{x_k-x_{k-1}\}$ converges to zero in norm. Taking this into account and taking the limit along k_j yield

$$(I-T)(x^\infty) = 0 \quad \Longleftrightarrow \quad T(x^\infty) = x^\infty.$$

Here the fact that $I-T$ is continuous and monotone is used. The proof of the uniqueness of the weak cluster point is standard, thus the details are omitted [6]. □

Corollary 3.1. *Since Algorithm 1 can be viewed as a special case of (1.1), with $\alpha_k = \alpha \in [0.80, 0.99]$, the statement "Choose $t_k \in [t_{k-1}, \bar{t}^+]$" in Algorithm 1 corresponds to (3.2) and (3.3) and $\sigma = 0.1$. As a direct consequence of Theorem 3.1, the sequence $\{x_k\}$ generated by Algorithm 1 converges weakly to a fixed point of T.*

Remark 3.1. Next, we numerically demonstrate assumption (3.3) to some extent. For brevity, we simply set

$$\alpha_k \equiv \alpha, \quad t_k \equiv t, \quad \eta(\sigma) := (1-\sigma)(\alpha^{-1}-1).$$

Then assumption (3.3) reduces to

$$t < \min\left\{0.5\frac{-1+\sqrt{1+4(\sigma^{-1}\eta(\sigma)+1)\eta(\sigma)}}{\sigma^{-1}\eta(\sigma)+1}, 1-(1+\eta(\sigma))^{-1}\right\} = f(\sigma). \quad (3.6)$$

We can further obtain that the following inequality holds

$$1-(1+\eta(\sigma))^{-1} \geq 0.5\frac{-1+\sqrt{1+4(\sigma^{-1}\eta(\sigma)+1)\eta(\sigma)}}{\sigma^{-1}\eta(\sigma)+1}.$$

From (3.6), we have

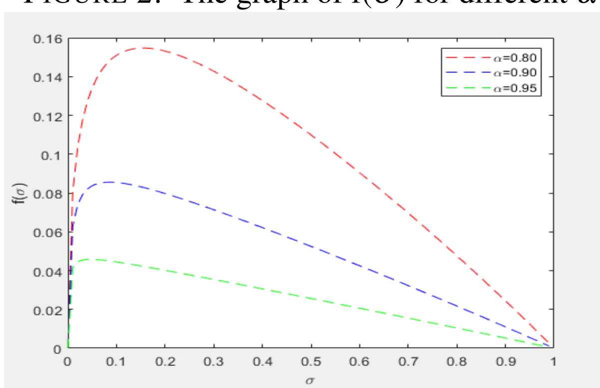
$$t < 0.5\frac{-1+\sqrt{1+4(\sigma^{-1}\eta(\sigma)+1)\eta(\sigma)}}{\sigma^{-1}\eta(\sigma)+1} := f(\sigma). \tag{3.7}$$

In contrast to (3.3), we no longer introduce the extra ε above because we turn to resort to Lemma 3.2. In addition, we have replaced $\leq$ there by $<$ here. As in Appendix, if $\alpha \in (0.5,1)$, then

$$f'(\sigma) < 0, \quad \sigma \in [0.5,1].$$

This means that $f(\sigma)$ is decreasing over $[0.5,1]$.

FIGURE 2. The graph of f(σ) for different α



Numerical demonstration of (3.7) is given in Table 1, where t_{new} stands for a slightly lower approximation of the f's maximum in (3.7) with respect to σ. Meanwhile, We also provide the values from [6, Table 2] for comparison and the corresponding data are given in Table 1, where $t(\alpha_k, \alpha_{k+1}, \varepsilon)$ is defined by (2.3).

TABLE 1. Numerical comparison between (3.7) and [6, Table 2]

α	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
t_{new}	0.304	0.275	0.246	0.216	0.186	0.155	0.121	0.086	0.046
σ	0.304	0.275	0.244	0.215	0.186	0.154	0.122	0.086	0.046
$t(\alpha_k, \alpha_{k+1}, \varepsilon)$	0.303	0.274	0.245	0.216	0.186	0.154	0.121	0.085	0.045

From Table 1, we can observe that our computed values of t_{new} are consistently larger than the corresponding values from [6, Table 2] for each sampling point.

Here we compare our results with the latest ones in [9]. Their key condition on inertial factors (see (2.4)) is

$$t_k(1+t_k)+(\alpha_k^{-1}-1)t_k(1-t_k) \leq (\alpha_{k-1}^{-1}-1)(1-t_{k-1})-\varepsilon,$$

where $\varepsilon \geq 0$. So in the cases of $t_k \equiv t$ and $\varepsilon = 0$, we can further obtain the upper bound of the inertial factors

$$\hat{t}(\alpha_k, \alpha_{k-1}) := 0.5 \frac{-(\alpha_k^{-1} + \alpha_{k-1}^{-1} - 1) + \sqrt{(\alpha_k^{-1} + \alpha_{k-1}^{-1} - 1)^2 + 4(2 - \alpha_k^{-1})(\alpha_{k-1}^{-1} - 1)}}{2 - \alpha_k^{-1}}.$$

As an instance, we take $\alpha_k = 0.90$ and list some comparison results in Table 2.

TABLE 2. Numerical comparison in the $\alpha_k = 0.90$ case

(α_k, α_{k-1})	(0.90,0.89)	(0.90,0.90)	(0.90,0.91)	(0.90,0.95)
$\hat{t}(\alpha_k, \alpha_{k-1})$	0.09377	0.08558	0.07734	0.04376
t_{new}	0.09379	0.08556	0.07735	0.04413

Remark 3.2. For the KM iteration, choosing α close to 1 in its accelerated and inertial versions [6] is generally a good strategy. In this case, it is noted that selecting σ to be equal to or close to 0.1 has been found to be a favorable choice; see Table 1 and Figure 2.

4. CONCLUSIONS

In this article, we considered the Krasnosel'skiĭ-Mann iteration for finding a fixed point of non-expansive operators in real Hilbert spaces. Our main contribution is that we proposed new inertial factors with desirable upper bounds better than or comparable to existing ones. It deserves pointing out that our proof is concise and self-contained.

APPENDIX

Here, we prove that if $\alpha \in (0.5, 1)$, then $f(\sigma)$ defined in (3.7) is decreasing over the interval $[0.5, 1]$. To this end, we need to prove that $f'(\sigma) < 0$, $\sigma \in [0.5, 1]$. Setting $s := \alpha^{-1} - 1$, one sees that $s \in (0, 1)$. It follows from (3.7) that

$$\begin{aligned} f(\sigma) &= 0.5 \frac{-1 + \sqrt{1 + 4\left((\sigma^{-1} - 1)(\alpha^{-1} - 1) + 1\right)(1 - \sigma)(\alpha^{-1} - 1)}}{(\sigma^{-1} - 1)(\alpha^{-1} - 1) + 1} \\ &= 0.5 \frac{-1 + \sqrt{1 + 4s\left(s\sigma^{-1} + (s - 1)\sigma - 2s + 1\right)}}{s\sigma^{-1} - s + 1}. \end{aligned}$$

Denote by $\Delta := 1 + 4s\left(s\sigma^{-1} + (s-1)\sigma - 2s + 1\right)$. Obviously, we have

$$2s - 1 \leq s\sigma^{-1} - (1 - s)\sigma \leq \frac{5}{2}s - \frac{1}{2},$$

which means $1 \leq \Delta \leq 2s^2 + 2s + 1$. Meanwhile, we also have

$$f(\sigma) = \frac{-0.5}{s\sigma^{-1} - s + 1} + \frac{0.5\sqrt{\Delta}}{s\sigma^{-1} - s + 1},$$

and its derivative $f'(\sigma)$ is

$$\begin{aligned}&\frac{-0.5s\sigma^{-2}}{(s\sigma^{-1}-s+1)^2}+\frac{s\frac{1}{\sqrt{\Delta}}\left(-s\sigma^{-2}+(s-1)\right)\left(s\sigma^{-1}-s+1\right)+0.5s\sigma^{-2}\sqrt{\Delta}}{(s\sigma^{-1}-s+1)^2}\\ =&\frac{-0.5s}{((1-s)\sigma+s)^2}+\frac{s\left((s-1)\sigma^2-s\right)((1-s)\sigma+s)+0.5s\sigma\Delta}{\sqrt{\Delta}\sigma((1-s)\sigma+s)^2}.\end{aligned}$$

Thus $f'(\sigma)\leq 0$ if and only if

$$s\left((s-1)\sigma^2-s\right)((1-s)\sigma+s)+0.5s\sigma\Delta\leq 0.5s\sigma\sqrt{\Delta},$$

which must hold provided that $s\left((s-1)\sigma^2-s\right)((1-s)\sigma+s)+0.5s\sigma\Delta\leq 0.5s\sigma$ due to $\Delta\geq 1$. Eliminating s on both sides yields

$$\left((s-1)\sigma^2-s\right)((1-s)\sigma+s)+2s^2+(2s^2-2s)\sigma^2+(2s-4s^2)\sigma\leq 0.$$

Next, we only need to prove that if $\sigma\in[0.5,1]$, then this inequality above is valid. In fact, by rearranging all terms, we can derive

$$-(s-1)^2\sigma^3+3s(s-1)\sigma^2+s(1-3s)\sigma+s^2\leq 0.$$

Setting

$$g(\sigma)=-(s-1)^2\sigma^3+3s(s-1)\sigma^2+s(1-3s)\sigma+s^2,$$

we see that

$$\begin{aligned}g'(\sigma)&=-3(s-1)^2\sigma^2+6s(s-1)\sigma+s(1-3s),\\ g''(\sigma)&=-6(s-1)^2\sigma+6s(s-1).\end{aligned}$$

Since $g''(\sigma)\leq 0$ whenever $s\in(0,1)$ and $\sigma\in[0.5,1]$, $g'(\sigma)$ is decreasing over the interval $[0.5,1]$. So, it is immediate that

$$g'(\sigma)\leq g'(0.5)=-\frac{3}{4}s^2-\frac{1}{2}s-\frac{3}{4}<0,\quad \sigma\in[0.5,1],$$

which implies that $g(\sigma)$ is decreasing over the interval $[0.5,1]$. Finally, we have

$$g(\sigma)\leq g(0.5)=\frac{1}{8}(s^2-1)<0,\quad \sigma\in[0.5,1].$$

The proof is complete.

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