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OPTIMALITY CONDITIONS AND DUALITY FOR E-DIFFERENTIABLE FRACTIONAL MULTIOBJECTIVE INTERVAL VALUED OPTIMIZATION PROBLEMS WITH E-INVEXITY

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Abstract. In this paper, a class of fractional multiobjective interval valued optimization problems with E-invexity is considered. First, the definition of the E-invex fractional interval value function is given under the interval order relation, and the existence of these fractional interval-valued functions is verified by examples. Second, we present the E-KKT necessary optimality conditions and the sufficient optimality conditions for a fractional interval-valued optimization problem (FIVP_E) under E-invexity. Last, the Mond-Weir E-dual problem (DFIVP_E) of (FIVP_E) is established, and several E-duality theorems are obtained under E-invexity. To some extent, this paper generalizes the existing relevant results obtained recently.

Keywords. Fractional interval valued optimization; LU-E-Pareto solution; LU-E-invex fraction interval value function; Mond-Weir E-dual.

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1. INTRODUCTION

The interval-valued optimization problems, which find numerical applications in the real world, are a class of optimization problems with uncertain parameters. In the interval mathematics [1], the interval is defined as a new type of number, i.e., the interval number. Ishibuchi and Tanaka [10] proposed the definition of LU, LC, UC, and CW-order relations for interval numbers. A number of results were investigated under these order relations. Wu [11] gave a new class of functions LU-convex interval-valued functions and derived KKT optimality conditions for interval-valued optimization problems. In recent years, optimality conditions and the applicability of duality revived interest in interval-valued optimization problems; see, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9] and the references therein.

Furthermore, convexity plays a key role in mathematical optimization and other related problems, particularly in the characterisation of optimality conditions. However, in real-world applications, many mathematical models of optimization problems do not easily satisfy the requirement of convexity. Thus, they usually consider convexity in the generalized sense. In general, generalized convexity includes E-convexity, G-convexity, B-convexity, pseudo convexity, invexity, and so on. The problems of generalized convex programming were investigated recently.

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Specifically, Youness [12] presented the definition of E-convex sets, E-convex functions, and obtained some new results for E-convex programming problems. Yang [13] corrected several erroneous conclusions in [5] and gave several examples. Fulga and Preda [14] gave a definition of E-preinvex functions and obtained several application results for E-preinvex nonlinear programming problems. Antczak and Abdulaleem [4] developed a class of E-differentiable non-convex vector optimization problems and investigated the E-KKT necessary optimality conditions and the sufficient optimality conditions for this optimization problem. Note that the study of interval-valued optimization problems is of continuing interest within generalized convexity. Van and Dinh [15] constructed Wolfe and Mond-Weir dual models for the interval-valued pseudoconvex optimization problem with equilibrium constraints, and also provided weak and strong duality theorems by using the notion of contingent epiderivatives with pseudoconvex functions in real Banach spaces. Guo et al. [16] worked on the construction of weak and strong dual theorems for interval-valued pseudoconvex optimization problems by using gH-symmetrically derivative, and derived generalized optimality conditions and duality theorems for interval-valued optimization problems with LU-convexity. In addition, some quantitative economic indicators were represented in some form of ratio. Recently, fractional optimization problems emerged. For the results in fractional multiobjective optimization, we refer to [17, 18] and the references therein. Debnath and Gupta [18] formulated KKT optimization conditions for fractional multiobjective interval-valued optimization problems by using the LU-V-invexity and LS-V-invexity assumptions. Liu [17] established necessary and sufficient conditions and derived a duality theorem for a class of non-smooth generalized minimax fractional programming problems with pseudo-invexity. To our knowledge, there is no result on E-invexity in fractional multiobjective interval-valued optimisation. A natural question arise: can we study optimality conditions and duality under the E-invexity assumption? This paper mainly addresses this question.

In this paper, a class of fractional multiobjective interval-valued optimization problems with E-invexity is considered. The structure of this paper is as follows. In Section 2, we give some basic definitions and present the definition of LU-E-invex fractional interval value function and verify this fractional interval-valued functions via an existence example. In Section 3, the E-differentiable fractional interval-valued optimization problem ($\mathrm{FIVP_E}$) is established. Under the E-invexity assumptions, simultaneously E-KKT necessary optimality conditions and the sufficient optimality conditions of the problem ($\mathrm{FIVP_E}$) are prove. In Section 4, the form of the Mond-Weir E-dual problem ($\mathrm{DFIVP_E}$) is given for problem ($\mathrm{FIVP_E}$) and several E-duality theorems are established. In Section 5, some concluding remarks and some directions for future research are given.

2. Preliminaries

From now on, R^n stands for the n-dimensional Euclidean space, X is an subset in R^n, R_n^+ means its nonnegative orthant, and $E : R^n \to R^n$ and $\eta : R^n \times R^n \to R^n$ are vector-valued mapping. For arbitrary $x = (x_1, x_2, \cdots, x_n)^T$ and $y = (y_1, y_2, \cdots, y_n)^T \in R^n$, we define

(i) $x = y \Leftrightarrow x_i = y_i, i = 1, 2, \cdots, n;$
(ii) $x > y \Leftrightarrow x_i > y_i, i = 1, 2, \cdots, n;$
(iii) $x \geqq y \Leftrightarrow x_i \geqq y_i, i = 1, 2, \cdots, n;$
(iv) $x \geqslant y \Leftrightarrow x \geqq y, and\ x \neq y.$

Let $\mathscr{I}$ be the family of sets composed by all closed intervals in R, with the set closed intervals $A=[a^L,a^U]$, where a^L and a^U represent the lower and upper bounds of A, respectively. If $a^L=a^U$, interval degenerates into real number, we can write $A=[a,a]=a, a\in R$. Let $A=[a^L,a^U]$ and $B=[b^L,b^U]$ be two closed intervals. Then

(i) $A+B=[a^L+b^L,a^U+b^U]$;
(ii) $-A=[-a^U,-a^L]$;
(iii) $A-B=[a^L-b^U,a^U-b^L]$;
(iv) $kA=\begin{cases}[ka^L,ka^U] & if \quad k\geq 0,\\ [ka^U,ka^L] & if \quad k<0;\end{cases}$
(v) $A\times B=[\min(a^Lb^L,a^Lb^U,a^Ub^L,a^Ub^U),\max(a^Lb^L,a^Lb^U,a^Ub^L,a^Ub^U)]$;
(vi) $A\div B=[a^L,a^U]\times[\frac{1}{b^U},\frac{1}{b^L}]$, where $b^L\neq 0$, $b^U\neq 0$.

In interval value optimization problems, order relations are usually used to compare intervals, and then a class of order relations is introduced

(i) $A\preceq_{LU} B$ if and only if $a^L\leq b^L$ and $a^U\leq b^U$;
(ii) $A\prec_{LU} B$ if and only if $A\preceq_{LU} B$ and $A\neq B$ hold, namely, one of the following formula holds

$$\begin{cases}a^L<b^L,\\ a^U\leq b^U,\end{cases} or \begin{cases}a^L\leq b^L,\\ a^U<b^U,\end{cases} or \begin{cases}a^L<b^L,\\ a^U<b^U.\end{cases}$$

Now, we introduce an other order relationship by Ishibuchi and Tanaka [10]. The interval $A=[a^L,a^U]$ also can be denoted by center and width as $A=\langle a^C,a^W\rangle$, where $a^C=\frac{1}{2}(a^L+a^U)$ is the center and $a^W=\frac{1}{2}(a^U-a^L)$ is the width of interval A. So, for minimization problems, we define the CW-order relationship as follows:

(i) $A\preceq_{CW} B$ iff $a^C\leq b^C$ and $a^W\leq b^W$;
(ii) $A\prec_{CW} B$ iff $A\preceq_{CW} B$ and $A\neq B$.

Similarly, the definition of UC-order relationship can also be obtained.

Lemma 2.1. [10] *For the closed intervals $A=[a^L,a^U]$ and $B=[b^L,b^U]$ in $\mathscr{I}$, it follows that*

(i) *$A\preceq_{UC} B$ iff $A\preceq_{LU} B$ or $A\preceq_{CW} B$;*
(ii) *$A\prec_{UC} B$ iff $A\prec_{LU} B$ or $A\prec_{CW} B$.*

In 2007, Wu [11] gave the definition of the interval value function. Recall that a function $f:R^n\to\mathscr{I}$ is called an interval value function, i.e., $f(x)=f(x_1,x_2,...,x_n)$ is a closed interval in $\mathscr{I}$ for each $x\in R^n$. It can also be noted as $f(x)=[f^L(x),f^U(x)]$, where $f^L(x)$ and $f^U(x)$ are real value functions defined on R^n with $f^L(x)\leq f^U(x)$. By the interval operation (vi), an interval valued function $\frac{f(x)}{g(x)}=\frac{[f^L(x),f^U(x)]}{[g^L(x),g^U(x)]}$ can be written as

$$\frac{f(x)}{g(x)}=[f^L(x),f^U(x)]\times\left[\frac{1}{g^U(x)},\frac{1}{g^L(x)}\right]=\left[\frac{f^L(x)}{g^U(x)},\frac{f^U(x)}{g^L(x)}\right],$$

where, $g^L(x)>0$, for all $x\in X$.

Let $\frac{A}{B}=\left(\frac{A_1}{B_1},\cdots,\frac{A_l}{B_l}\right)$ and $\frac{C}{D}=\left(\frac{C_1}{D_1},\cdots,\frac{C_l}{D_l}\right)$ be two fractional interval valued vectors, where $A_k,B_k,C_k,D_k\in\mathscr{I}, D_k\in\mathscr{I}^+$, $k=1,2,\cdots,l$, where $\mathscr{I}^+$ denotes the positive interval. Then

(i) $\frac{A}{B}\preceq_{LU}\frac{C}{D}$ iff $\frac{A_k}{B_k}\preceq_{LU}\frac{C_k}{D_k}$, for all $k=1,2,\cdots,l$;

(ii) $\frac{A}{B} \prec_{LU} \frac{C}{D}$ iff $\frac{A_k}{B_k} \preceq_{LU} \frac{C_k}{D_k}$, $k = 1,2,\cdots,l$, and $\frac{A_j}{B_j} \prec_{LU} \frac{C_j}{D_j}$ for least one $j \in 1,2,\cdots,l$.

The definition of weak differentiability of the interval value function introduced by Wu [11] and the notion of E-differentiability recalled by Megahed et al. [19].

Definition 2.1. [11] Let X be an open set in R^n. One says that the interval value function $f(x) = [f^L(x), f^U(x)]$ is weakly differentiable at $x_0 \in X$ if the real value function f^L and f^U are differentiable at x_0.

Definition 2.2. [19] Let $f : R^n \to I$ be a differentiable real valued function, $x_0 \in R^n$, and $E : R^n \to R^n$. One says that f is E-differentiable at x_0 if and only if the composite function $f \circ E$ is a differentiable function, that is,

$$(f\circ E)(x) = (f\circ E)(x_0) + \nabla(f\circ E)(x_0)(x-x_0) + \theta(x_0, x-x_0)\,\|x-x_0\|,$$

where $x \to x_0$ as $\theta(x_0, x-x_0) \to 0$.

Definition 2.3. [4] Let $E : R^n \to R^n$ be a vector value mapping. One says that the interval value function, $f : R^n \to \mathscr{I}$, $f(x) = [f^L(x), f^U(x)]$ is E-differentiable at $x_0 \in S$ if and only if the real value functions $f^L \circ E$ and $f^U \circ E$ are differentiable functions, that is,

$$(f^L\circ E)(x) = (f^L\circ E)(x_0) + \nabla(f^L\circ E)(x_0)(x-x_0) + \theta^L(x_0, x-x_0)\,\|x-x_0\|$$

and

$$(f^U\circ E)(x) = (f^U\circ E)(x_0) + \nabla(f^U\circ E)(x_0)(x-x_0) + \theta^U(x_0, x-x_0)\,\|x-x_0\|,$$

where $x \to x_0$ as $\theta^L(x_0, x-x_0) \to 0$ and $\theta^U(x_0, x-x_0) \to 0$.

Next, we recall the definition of E-invex function proposed by Fulga and Preda [14].

Definition 2.4. [14] Let X be an open set in R^n. Let $\eta : X \times X \to X$ and $E : X \to X$ be both vector value mapping. E-differentiable real valued function $f : X \to R$ is said to be (strictly) E-invex about η at $y \in X$ if, for any $x \in X$ and $0 \le \lambda \le 1$, $f(E(x)) - f(E(y))(>)\geq \eta^T(E(x), E(y))\nabla f(E(y))$.

Now, we introduce the concept of the LU-E-invex interval valued function.

Definition 2.5. Let X be an open set in R^n, and let $\eta : X \times X \to X$ and $E : X \to X$ be both vector value mappings. For $f = [f^L(x), f^U(x)]$ and $g = [g^L(x), g^U(x)]$, E-differentiable fractional interval valued function $\frac{f}{g}(x) = \left[\frac{f^L}{g^U}(x), \frac{f^U}{g^L}(x)\right]$ $(g^L > 0)$ is said to be a (strictly) LU-E-invex fractional interval value function about η at $y \in X$ if $\frac{f^U}{g^L}(x)$ and $\frac{f^L}{g^U}(x)$ are (strictly) E-invex about the same η at X.

The following example illustrates the existence of the LU-E-invex fractional interval valued function.

Example 2.1. Let $A = (0, \frac{\pi}{2})$, $E : A \to A$ be a mapping, and $\eta : A \times A \to A$ be an mapping.

$$E(x) = \frac{\pi}{2} - x,$$

$$\eta(x,y) = \begin{cases} \frac{\sin x - \sin y}{\cos y}, & if \quad x > y, \\ 0, & if \quad x = y, \\ \frac{2(\sin x - \sin y)}{\cos y}, & if \quad x < y. \end{cases}$$

Let interval valued function $f(x) = [\sin x, 2\sin x]$ and $g(x) = [\frac{1}{2}\tan x, \tan x]$. It can be seen that $\frac{f}{g}$ is LU-E-invex on A. Now we verify this conclusion.

By interval operation, we can obtain $\frac{f(E(x))}{g(E(x))} = [\sin x, 4\sin x]$.

Case 1: for $x > y$, we have

$$\frac{f^L(E(x))}{g^U(E(x))} - \frac{f^L(E(y))}{g^U(E(y))} = \cos y \frac{\sin x - \sin y}{\cos y} = \nabla \frac{f^L}{g^U}(E(y))\eta^T(E(x), E(y)).$$

Case 2: for $x = y$, we have

$$\frac{f^L(E(x))}{g^U(E(x))} - \frac{f^L(E(y))}{g^U(E(y))} = \sin x - \sin y \geq \nabla \frac{f^L}{g^U}(E(y))\eta^T(E(x), E(y)).$$

Case 3: for $x < y$, we have

$$\frac{f^L(E(x))}{g^U(E(x))} - \frac{f^L(E(y))}{g^U(E(y))} = \sin x - \sin y \geq \cos y \frac{2\sin x - 2\sin y}{\cos y} = \nabla \frac{f^L}{g^U}(E(y))\eta^T(E(x), E(y)).$$

Hence, for all $x, y \in A$, we have

$$\frac{f^L(E(x))}{g^U(E(x))} - \frac{f^L(E(y))}{g^U(E(y))} \geq \nabla \frac{f^L(E(y))}{g^U(E(y))}\eta^T(E(x), E(y)).$$

Similarly,

$$\frac{f^U(E(x))}{g^L(E(x))} - \frac{f^U(E(y))}{g^L(E(y))} \geq \nabla \frac{f^U(E(y))}{g^L(E(y))}\eta^T(E(x), E(y)),$$

which means that $\frac{f}{g}$ is LU-E-invex on A.

Next, we now introduce the E-invexity of the fractional interval valued functions.

Definition 2.6. Let $\frac{F}{G} = \left(\frac{f_1}{g_1}, \cdots, \frac{f_l}{g_l}\right)$, where $f_k = [f_k^L(x), f_k^U(x)]$, $g_k = [g_k^L(x), g_k^U(x)]$, $g_k^L > 0$, $k = 1, \cdots, 1$, be a vector fractional interval valued function. Then

(i) $\frac{F}{G}$ is LU-E-invex (strictly LU-E-invex) if and only if $\frac{f_k}{g_k}$ are LU-E-invex (strictly LU-E-invex) for all $k = 1, \cdots, l$;

(ii) $\frac{F}{G}$ is UC-E-invex (strictly UC-E-invex) if and only if $\frac{f_k}{g_k}$ are UC-E-invex (strictly UC-E-invex) for all $k = 1, \cdots, l$;

(iii) $\frac{F}{G}$ is CW-E-invex (strictly CW-E-invex) if and only if $\frac{f_k}{g_k}$ are CW-E-invex (strictly CW-E-invex) for all $k = 1, \cdots, l$.

Remark 2.2. From Lamma 2.1, we can see that $\frac{F}{G} = \left(\frac{f_1}{g_1}, \cdots, \frac{f_l}{g_l}\right)$ is UC-E-invex if and only if LU-E-invex or CW-E-invex.

3. Optimality Conditions for (FIVP$_E$)

In this section, we consider the nondifferentiable fractional interval valued multiobjective optimization problem as follow:

$$(FIVP_E)\begin{cases} \min & \frac{F(E(x))}{G(E(x))} = \left(\frac{f_1(E(x))}{g_1(E(x))}, \cdots, \frac{f_l(E(x))}{g_l(E(x))}\right) \\ s.t. & h_j(E(x)) \leq 0, j = 1, \cdots, m, \\ & x \in X, \end{cases}$$

where X is a open subset of R^n, $f_k(E(x)) = [f_k^L(E(x)), f_k^U(E(x))]$, and $g_k(E(x)) = [g_k^L(E(x)), g_k^U(E(x))]$, $k = 1,2,\cdots,l$ are interval valued functions. The fractional interval valued vector function $\frac{F(x)}{G(x)}$ need not a differentiable interval valued function, but a E-differentiable fractional interval valued function. It is assumed that $g_k^L(E(x)) > 0$ for all $x \in X$ and $k = 1,2,\cdots,l$. Let

$$X_E = \left\{x \in X | h_j(E(x)) \leq 0, j = 1,\cdots,m\right\}$$

be the feasible set of problem ($\mathrm{FIVP_E}$). For the feasible point $x^* \in X_E$, let

$$J(E(x^*)) = \{j \in \{1,\cdots,m\} | h_j(E(x^*)) = 0\}$$

be the index set of the active constraints for the feasible set X_E.

For such nondifferentiable fractional interval valued multiobjective optimization problem, we now introduce the following concepts.

Definition 3.1. Let x^* be a feasible solution of the problem $(FIVP_E)$. We say that x^* is a LU-E-Pareto optimal solution to problem $(FIVP_E)$ if and only if there exists no $x \in X_E$ such that $\frac{F(E(x))}{G(E(x))} \prec_{LU} \frac{F(E(x^*))}{G(E(x^*))}$.

Definition 3.2. Let x^* be a feasible solution to problem $(FIVP_E)$. We say that x^* is a strongly LU-E-Pareto optimal solution to problem $(FIVP_E)$ if and only if there exists no $x \in X_E$ such that $\frac{F(E(x))}{G(E(x))} \preceq_{LU} \frac{F(E(x^*))}{G(E(x^*))}$.

Definition 3.3. Let x^* be a feasible solution to problem $(FIVP_E)$. We say that x^* is a weakly LU-E-Pareto optimal solution to problem $(FIVP_E)$ if and only if there exists no $x \in X_E$ such that $\frac{f_k(E(x))}{g_k(E(x))} \prec_{LU} \frac{f_k(E(x^*))}{g_k(E(x^*))}$ for all $k = 1,\cdots,l$.

Remark 3.1. Let O, P, and Q be the strongly LU-E-Pareto optimal solution set, LU-E-Pareto optimal solution set, and weakly LU-E-Pareto optimal solution set, respectively. Then $O \subseteq P \subseteq Q$.

Constraint qualification is an important guarantee for the optimality condition in generalized convex optimization problems. Therefore, in order to prove the necessary condition of the solutions of problem ($\mathrm{FIVP_E}$), a concept of constraint qualification is introduced below.

Definition 3.4. Let X_E be a nonempty feasible set and $x_0 \in clX_E$ (the closure of X_E). $d \neq 0$ is called a feasible direction at $x_0 \in X_E$ if there exists a $\delta > 0$, such that $x_0 + \theta d \in X_E$ for all $\theta \in (0,\delta)$. In problem ($\mathrm{FIVP_E}$), if $d^T \nabla \frac{f_k^L}{g_k^U}(E(x_0)) < 0$, $d^T \nabla \frac{f_k^U}{g_k^L}(E(x_0)) < 0$, $k = 1,\cdots,l$, d is also called a feasible descent direction at $x_0 \in X_E$.

Remark 3.2. There is no feasible descent direction for the LU-E-Pareto optimal solution of ($\mathrm{FIVP_E}$). In fact, we suppose that x^* is a LU-E-Pareto optimal solution to problem ($\mathrm{FIVP_E}$). If there exists a vector $d \in R^n$ satisfying $d^T \nabla \frac{f_k^L}{g_k^U}(E(x^*)) < 0$, $d^T \nabla \frac{f_k^U}{g_k^L}(E(x^*)) < 0$, $k = 1,\cdots,l$, by E-differentiable, for $\alpha > 0$, there exist

$$\frac{f_k^L}{g_k^U}(E(x^*) + \alpha d) = \frac{f_k^L}{g_k^U}(E(x^*)) + \alpha d^T \nabla \frac{f_k^L}{g_k^U}(E(x^*)) + \theta^L(x^*\text{f-}\alpha d) \|\alpha d\| < \frac{f^L}{g^U}(E(x^*)),$$

and

$$\frac{f_k^U}{g_k^L}(E(x^*)+\alpha d) = \frac{f_k^U}{g_k^L}(E(x^*)) + \alpha d^T \nabla \frac{f_k^U}{g_k^L}(E(x^*)) + \theta^L(x^*\text{f-}\alpha d)\|\alpha d\| < \frac{f_k^U}{g_k^L}(E(x^*)).$$

Thus

$$\frac{f_k^L}{g_k^U}(E(x^*)+\alpha d) < \frac{f_k^L}{g_k^U}(E(x^*)),\ \frac{f_k^U}{g_k^L}(E(x^*)+\alpha d) < \frac{f_k^U}{g_k^L}(E(x^*)).$$

It follows that

$$\frac{F(E(x^*+\alpha d))}{G(E(x^*+\alpha d))} < \frac{F(E(x^*))}{G(E(x^*))}.$$

and we find a better point than point x^*. This contradicts that x^* is a LU-E-Pareto optimal solution.

Definition 3.5. If $h_j, j \in J(E(x^*))$ is E-invex with respect to same η at x^*, and there exists $\hat{x} \in X$ such that $h_j(E(\hat{x})) < 0$ for all $j \in J(E(x^*))$, we called the E-constraint qualification (ECQ) holds at x^*.

The classic Tucker's theorem of the alternative states is as follows. Let A be a $m \times n$ matrice, and let C be a matrix. Exactly one of the following systems has a solution:

System 1: $Ax \leq 0, Ax \neq 0$, and $Cx \leq 0$ for some $x \in R^n$;

System 2: $A^T\lambda + C^T\mu = 0$ for some (λ, μ), $\lambda > 0$, and $\mu \geq 0$.

Theorem 3.1. *(E-KKT Necessary Conditions) Suppose that $\frac{F}{G}$ is a E-differentiable fractional interval value function at $x^* \in X_E$, and $h_j,\ j = 1,\cdots,m$ are E-invex with respect to same η at $x^* \in X_E$. Furthermore, suppose that the (ECQ) holds at x^*. If x^* is a LU-E-Pareto optimal solution to problem $(FIVP_E)$, then there exist multipliers λ_k^L, $\lambda_k^U \geq 0$, $k = 1, \cdots, l$, and $\mu_j \geq 0$, $j = 1, ..., m$, such that*

$$\sum_{k=1}^{l} \lambda_k^L \nabla \frac{f_k^L}{g_k^U}(E(x^*)) + \sum_{k=1}^{l} \lambda_k^U \nabla \frac{f_k^U}{g_k^L}(E(x^*)) + \sum_{j=1}^{m} \mu_j \nabla h_j(E(x^*)) = 0, \tag{3.1}$$

$$\mu_i h_j(E(x^*)) = 0, j = 1, ..., m. \tag{3.2}$$

Proof. We are going to prove it by contradiction. Now, suppose that there exists a vector $d^* \in R^n$ that satisfies

$$\nabla \frac{f_k^L}{g_k^U}(E(x^*))^T d^* < 0, \nabla \frac{f_k^U}{g_k^L}(E(x^*))^T d^* < 0, \nabla h_j(E(x^*))^T d^* \leq 0, j \in J(E(x^*)).$$

Since the (ECQ) is satisfied at x^*, then there exists $x \in X$ such that $h_j(E(x)) < 0 = h_j(E(x^*))$ for all $j \in J(E(x^*))$. Since $h_j(j \in J(E(x^*)))$ are E-invex at x^*, we have

$$\eta(E(x), E(x^*))^T \nabla h_j(E(x^*)) \leq h_j(E(x)) - h_j(E(x^*)) < 0, \forall j \in J(E(x^*)).$$

Then, $d = \eta(E(x), E(x^*))$ is a feasible direction at x^*. Let $\hat{d} = d^* + (1/n)d$. Ten $\nabla h_j(E(x^*))^T \hat{d} \leq 0, j \in J(E(x^*))$. Therefore, $\hat{d}$ is a feasible direction at x^*. For sufficiently large n, we can also obtain $\nabla \frac{f^L}{g^U}(E(x^*))^T \hat{d} < 0, \nabla \frac{f^U}{g^L}(E(x^*))^T \hat{d} < 0$, that is, $\nabla \frac{F}{G}(E(x^*))^T \hat{d} \prec [0,0]$. Hence, $\hat{d}$ is a feasible descent direction at x^*. However, one obtains form Remak 3.2 the contrary to the assumption that x^* is a LU-E-Pareto optimal solution.

According to the Tucker's theorem, there exists multipliers $\lambda_k^L, \lambda_k^U \in R, k = 1, \cdots, l$, and $0 \le \mu_i \in R, i = 1, \ldots, m$, such that

$$\sum_{k=1}^{l} \lambda_k^L \nabla \frac{f_k^L}{g_k^U}(E(x^*)) + \sum_{k=1}^{l} \lambda_k^U \nabla \frac{f_k^U}{g_k^L}(E(x^*)) + \sum_{j \in J(E(x^*))} \mu_j \nabla h_j(E(x^*)) = 0.$$

Furthermore, letting $\mu_j = 0, j \in \{1, \ldots, m\} \backslash J(E(x^*))$, we have

$$\sum_{k=1}^{l} \lambda_k^L \nabla \frac{f_k^L}{g_k^U}(E(x^*)) + \sum_{k=1}^{l} \lambda_k^U \nabla \frac{f_k^U}{g_k^L}(E(x^*)) + \sum_{j=1}^{m} \mu_j \nabla h_j(E(x^*)) = 0,$$

$$\mu_j h_j(E(x^*)) = 0, i = 1, \ldots, m,$$

which completes the proof. □

Theorem 3.2. *(E-KKT Sufficient Conditions) Let x^* be a feasible solution to problem $(FIVP_E)$, and $\frac{F}{G}$ be LU-E-invex with respect to η at x^*. Let $h_j, j = 1, \ldots, m$ be E-differentiable and E-invex with respect same η at x^*. If there exist multipliers $\lambda_k^L, \lambda_k^U \ge 0$, $\mu_j \ge 0$, $j = 1, \ldots, m$ with (3.1)-(3.2), then x^* is a LU-E-Pareto optimal solution to $(FIVP_E)$.*

Proof. On the contrary, if x^* is not a LU-E-Pareto optimal solution to ($\mathrm{FIVP_E}$), then there exist $\bar{x} \in X_E$ such that

$$\frac{F(E(\bar{x}))}{G(E(\bar{x}))} \prec_{LU} \frac{F(E(x^*))}{G(E(x^*))}, \tag{3.3}$$

which implies, for all $1 \le k \le l$,

$$\begin{cases} \frac{f_k^L}{g_k^U}(E(\bar{x})) \le \frac{f_k^L}{g_k^U}(E(x^*)), \\ \frac{f_k^U}{g_k^L}(E(\bar{x})) < \frac{f_k^U}{g_k^L}(E(x^*)), \end{cases} \text{or} \begin{cases} \frac{f_k^L}{g_k^U}(E(\bar{x})) < \frac{f_k^L}{g_k^U}(E(x^*)), \\ \frac{f_k^U}{g_k^L}(E(\bar{x})) \le \frac{f_k^U}{g_k^L}(E(x^*)), \end{cases} \text{or} \begin{cases} \frac{f_k^L}{g_k^U}(E(\bar{x})) < \frac{f_k^L}{g_k^U}(E(x^*)), \\ \frac{f_k^U}{g_k^L}(E(\bar{x})) < \frac{f_k^U}{g_k^L}(E(x^*)). \end{cases}$$

For suitable $\lambda_k^L, \lambda_k^U \in R^+$, $k = 1, \cdots, l$, we have

$$\sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(\bar{x})) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(\bar{x})) < \sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(x^*)) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(x^*)).$$

Since $\frac{F}{G}$ is E-invex at x^*, we see that

$$0 > \frac{f_k^L}{g_k^U}(E(\bar{x})) - \frac{f_k^L}{g_k^U}(E(x^*)) \ge \eta(E(\bar{x}), E(x^*))^T \nabla \frac{f_k^L}{g_k^U}(E(x^*)),$$

and

$$0 > \frac{f_k^U}{g_k^L}(E(\bar{x})) - \frac{f_k^U}{g_k^L}(E(x^*)) \ge \eta(E(\bar{x}), E(x^*))^T \nabla \frac{f_k^U}{g_k^L}(E(x^*)).$$

Multiplying both sides of this equation by $\lambda_k^L, \lambda_k^U \in R^+$, $k = 1, \cdots, l$, we have

$$0 > \sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(\bar{x})) - \sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(x^*)) \ge \sum_{k=1}^{l} \lambda_k^L \eta(E(\bar{x}), E(x^*))^T \nabla \frac{f_k^L}{g_k^U}(E(x^*)), \tag{3.4}$$

and

$$0 > \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(\bar{x})) - \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(x^*)) \ge \sum_{k=1}^{l} \lambda_k^U \eta(E(\bar{x}), E(x^*))^T \nabla \frac{f_k^U}{g_k^L}(E(x^*)). \tag{3.5}$$

Similarly, since $h_j, j=1,\cdots,m$ are E-invex at x^*, we see that

$$h_j(E(\bar{x})) - h_j(E(x^*)) \geq \eta(E(\bar{x}),E(x^*))^T \nabla h_j(E(x^*)).$$

From (3.2), for suitable $\mu_j, j=1,\cdots,m$, we have

$$\sum_{j=1}^{m} \mu_j h_j(E(\bar{x})) \geq \sum_{j=1}^{m} \mu_j \eta(E(\bar{x}),E(x^*))^T \nabla h_j(E(x^*)). \tag{3.6}$$

Now, from (3.4), (3.5), and (3.6), we can obtain

$$\begin{aligned}
&\sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(\bar{x})) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(\bar{x})) + \sum_{j=1}^{m} \mu_j h_j(E(\bar{x})) \\
&\geq \sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(x^*)) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(x^*)) \\
&\quad + \eta(E(\bar{x}),E(x^*))^T \left(\sum_{k=1}^{l} \lambda_k^U \nabla \frac{f_k^U}{g_k^L}(E(x^*)) + \sum_{k=1}^{l} \lambda_k^L \nabla \frac{f_k^L}{g_k^U}(E(x^*)) + \sum_{j=1}^{m} \mu_j \nabla h_j(E(x^*)) \right).
\end{aligned} \tag{3.7}$$

It follows from (3.1) that

$$\begin{aligned}
&\sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(\bar{x})) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(\bar{x})) + \sum_{j=1}^{m} \mu_j h_j(E(\bar{x})) \\
&\geq \sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(x^*)) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(x^*)).
\end{aligned} \tag{3.8}$$

Since $\bar{x} \in X_E$, $h_j(E(\bar{x})) \leq 0$, (3.8) implies that

$$\sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(\bar{x})) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(\bar{x})) \geq \sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(x^*)) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(x^*)),$$

which contradicts (3.3), so that x^* is really a LU-E-Pareto optimal solution to ($\mathrm{FIVP_E}$). This completes the proof. □

Now, we now illustrate the validity of Theorem 3.2.

Example 3.1. Consider the nondifferentiable optimization problem with the multiple fractional interval valued objective function given by

$$(P_1) \begin{cases} \min & \frac{F(E(x))}{G(E(x))} = \left(\frac{f_1(E(x))}{g_1(E(x))}, \frac{f_2(E(x))}{g_2(E(x))} \right) \\ s.t. & h_j(E(x)) \leq 0, j=1,2, \\ & x \in R, \end{cases}$$

where

$$\frac{f_1(x)}{g_1(x)} = \frac{\left[\sqrt[3]{x^2}+1, \sqrt[3]{x^2}+2\right]}{\left[\sqrt[3]{x}+2, \sqrt[3]{x}+3\right]}, \frac{f_2(x)}{g_2(x)} = \frac{\left[\sqrt[3]{x^2}+\sqrt[3]{x^4}, 2\left(\sqrt[3]{x^2}+\sqrt[3]{x^4}\right)\right]}{\left[\sqrt[3]{x^2}+1, \sqrt[3]{x^2}+2\right]},$$

$$h_1(x) = \sqrt[3]{x} - 2,\ h_2(x) = -\sqrt[3]{x} + 1.$$

For $E(x) = x^3$, one has

$$\frac{f_1(E(x))}{g_1(E(x))} = \frac{\left[x^2+1, x^2+2\right]}{[x+2, x+3]}, \frac{f_2(E(x))}{g_2(E(x))} = \frac{\left[x^2+x^4, 2\left(x^2+x^4\right)\right]}{[x^2+1, x^2+2]},$$

$$h_1(E(x)) = x-2, \; h_2(E(x)) = -x+1.$$

Let $\eta(x,y) = x-y$. It is easy to see that the above function satisfies the hypothesis of Theorem 3.2 at $x=1$, i.e. $\frac{F(E(x))}{G(E(x))}$ is LU-E-invex about $\eta(x,y) = x-y$, and $h_j(E(x)) \leq 0, j=1,2$, is LU-E-invex about the same $\eta(x,y) = x-y$. There exist $\lambda_1^L = \frac{1}{2}$, $\lambda_1^U = \frac{3}{4}$, $\lambda_2^L = \frac{9}{28}$, $\lambda_2^U = \frac{1}{64}$, $\mu_1 = 0$, and $\mu_2 = 1$ satisfy the formula (3.1)-(3.2) at $x=1$. Therefore, $x=1$ is a LU-E-Pareto optimal solution of problem (P_1), that is, Theorem 3.2 is true.

4. Mond-Weir E-Duality Theorem for $(\mathrm{FIVP_E})$

In this section, we consider the primal and dual fractional interval valued problem. The Mond-Weir dual problems are discussed. We formulate a parametric dual problem in the sense of Mond-Weir for the fractional interval vauled optimization problem $(\mathrm{FIVP_E})$ as follows:

$$(DFIVP_E)\begin{cases} \max & \frac{F(E(y))}{G(E(y))} = \left(\frac{f_1(E(x))}{g_1(E(x))}, \cdots, \frac{f_l(E(x))}{g_l(E(x))}\right) \\ s.t. & \sum_{k=1}^{l} \lambda_k^L \nabla \frac{f_k^L}{g_k^U}(E(y)) + \sum_{k=1}^{l} \lambda_k^U \nabla \frac{f_k^U}{g_k^L}(E(y)) + \sum_{j=1}^{m} \mu_j \nabla h_j(E(y)) = 0, \\ & \mu_j h_j(E(y)) \geq 0, j=1,\cdots,m, \\ & \mu_j \geq 0, j=1,\cdots,m, \end{cases}$$

where $f_k(E(x)) = \left[f_k^L(E(x)), f_k^U(E(x))\right]$, and $g_k(E(x)) = \left[g_k^L(E(x)), g_k^U(E(x))\right]$, $k=1,2,\cdots,l$ are interval valued functions ,and X is a subset of R^n. It is assumed that $g_k^L(E(x)) > 0$ for all $x \in X$ and $k=1,2,\cdots,l$, λ_k^L, $\lambda_k^U \geq 0$, and $\mu \geq 0$. Let

$$D_E = \Bigg\{ (y, \lambda^L, \lambda^U, \mu) \in R^n \times R_+^l \times R_+^l \times R_+^m \,\Bigg|\, \sum_{k=1}^{l} \lambda_k^L \nabla \frac{f^L}{g^U}(E(y)) + \sum_{k=1}^{l} \lambda_k^U \nabla \frac{f^U}{g^L}(E(y)) + \sum_{j=1}^{m} \mu_j \nabla h_j(E(y)) = 0, \mu_j h_j(E(y)) \geq 0, \mu_j \geq 0, j=1,\cdots,m \Bigg\}$$

be the feasible set of $(\mathrm{DFIVP_E})$.

Definition 4.1. Let $(y^*, \lambda^L, \lambda^U, \mu)$ be a feasible solution to problem $(DFIVP_E)$. We say that $(y^*, \lambda^L, \lambda^U, \mu)$ is a LU-E-Pareto optimal solutionto problem $(DFIVP_E)$ if there no exists $(y, \lambda^L, \lambda^U, \mu) \in D_E$ such that $\frac{F(E(y^*))}{G(E(y^*))} \prec_{LU} \frac{F(E(y))}{G(E(y))}$.

Theorem 4.1. *(Weakly E-Duality) Let x be a feasible solution to $(FIVP_E)$, and let $(y, \lambda^L, \lambda^U, \mu)$ be a solutionto $(DFIVP_E)$. Further, assume that the following hypotheses are satisfied:*
(i) $\frac{F}{G}$ is LU-E-invex with respect to η at $y \in X_E \cup D_E$;
(ii) $h_j, j=1,\ldots,m$ are E-invex with respect same η at $y \in X_E \cup D_E$,
then $\frac{F(E(x))}{G(E(x))} \prec_{LU} \frac{F(E(y))}{G(E(y))}$ does not hold.

Proof. We assume that $\frac{F(E(x))}{G(E(x))} \prec_{LU} \frac{F(E(y))}{G(E(y))}$ is true, that is, for all $1 \le k \le l$,

$$\begin{cases} \frac{f_k^L}{g_k^U}(E(x)) \le \frac{f_k^L}{g_k^U}(E(y)), \\ \frac{f_k^U}{g_k^L}(E(x)) < \frac{f_k^U}{g_k^L}(E(y)), \end{cases} \text{or} \begin{cases} \frac{f_k^L}{g_k^U}(E(x)) < \frac{f_k^L}{g_k^U}(E(y)), \\ \frac{f_k^U}{g_k^L}(E(x)) \le \frac{f_k^U}{g_k^L}(E(y)), \end{cases} \text{or} \begin{cases} \frac{f_k^L}{g_k^U}(E(x)) < \frac{f_k^L}{g_k^U}(E(y)), \\ \frac{f_k^U}{g_k^L}(E(x)) < \frac{f_k^U}{g_k^L}(E(y)). \end{cases}$$

For suitable $\lambda_k^L, \lambda_k^U \in R^+$, $k = 1, \cdots, l$, we have

$$\sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(x)) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(x)) < \sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(y)) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(y)). \tag{4.1}$$

Observe that x is a feasible solution to (FIVP$_\mathrm{E}$), and $(y, \lambda^L, \lambda^U, \mu)$ is a feasible solution to (DFIVP$_\mathrm{E}$). Since $\frac{F}{G}$ is *LU*-*E*-invex at $y \in X_E \cup D_E$, by the Definition 2.1, for $k = 1, \cdots, l$, we see that

$$\frac{f_k^L}{g_k^U}(E(z)) - \frac{f_k^L}{g_k^U}(E(y)) \ge \nabla \frac{f_k^L}{g_k^U}(E(y)) \eta\left((E(z)), (E(y))\right),$$

$$\frac{f_k^U}{g_k^L}(E(z)) - \frac{f_k^U}{g_k^L}(E(y)) \ge \nabla \frac{f_k^U}{g_k^L}(E(y)) \eta\left((E(z)), (E(y))\right),$$

and

$$h_j(E(z)) - h_j(E(y)) \ge \nabla h_j(E(y)) \eta\left((E(z)), (E(y))\right), j = 1, \cdots, m,$$

where $z \in X_E \cup D_E$. Hence, they are also satisfied for $z = x \in X_E$.

Next, for λ^L, $\lambda^U \in R_+^l$, $\mu \in R_+^m$, we have

$$\begin{aligned} &\sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(x)) - \sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(y)) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(x)) - \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(y)) \\ &+ \sum_{j=1}^{m} \mu_j h_j(E(x)) - \sum_{j=1}^{m} \mu_j h_j(E(y)) \\ \ge &\left(\sum_{k=1}^{l} \lambda_k^L \nabla \frac{f_k^L}{g_k^U}(E(y)) + \sum_{k=1}^{l} \lambda_k^U \nabla \frac{f_k^U}{g_k^L}(E(y)) + \sum_{j=1}^{m} \mu_j \nabla h_j(E(y)) \right) \eta\left((E(x)), (E(y))\right). \end{aligned}$$

From the constraint of (DFIVP$_\mathrm{E}$), by $x \in X_E$ and $(y, \lambda^L, \lambda^U, \mu) \in D_E$, we obtain

$$\sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(x)) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(x)) \ge \sum_{k=1}^{l} \lambda_k^L \frac{f_k^L}{g_k^U}(E(y)) + \sum_{k=1}^{l} \lambda_k^U \frac{f_k^U}{g_k^L}(E(y)),$$

which contradicts (4.1). This completes the proof. □

Theorem 4.2. *(Strong E-Duality) Let x^* be a LU-E-Pareto optimal solution to $(FIVP_E)$, and let (ECQ) hold at x^*. Then, there exist $0 < \bar{\lambda}^L, \bar{\lambda}^U \in R^l$ such that $(x^*, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a feasible solution to $(DFIVP_E)$. If the assumptions of Theorem 4.1 are satisfied, then $(x^*, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a LU-E-Pareto optimal solution to $(DFIVP_E)$.*

Proof. From Theorem 3.1, we see that there exists $0 < \bar{\lambda}^L, \bar{\lambda}^U \in R^l, \bar{\mu} \in R^m$ such that $(x^*, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a feasible solution to (DFIVP$_\mathrm{E}$). From Theorem 4.1, we have that $(x^*, \bar{\lambda}^L, \bar{\lambda}^U, \bar{\mu})$ is a *LU*-*E*-Pareto optimal solution to (DFIVP$_\mathrm{E}$). This completes the proof. □

Theorem 4.3. *(Converse E-Duality) Let $(y^*,\lambda^L,\lambda^U,\mu)$ be a LU-E-Pareto optimal solution to $(DFIVP_E)$, and let the constraint qualification hold at y^*. Further, we assume that the objective function is a strict E-invex interval valued function, and the constraint function is strict LU-E-invex. Then y^* is an LU-E-Pareto optimal solution to $(FIVP_E)$.*

Proof. From Theorem 4.1, we can obtain the desired result easily. □

Theorem 4.4. *(Strict Converse E-duality) Let x^* and $(y^*,\lambda^L,\lambda^U,\mu)$ be an LU-E-Pareto optimal solution to problem $(FIVP_E)$ and problem $(DFIVP_E)$, respectively. If the object function $\frac{F}{G}$ is strict E-invex at $y^* \in X_E \cup D_E$, and constraint function $h_j(E(x)), j=1,\cdots,m$, are E-invex at y^*, then $x^*=y^*$.*

Proof. Conversely, we suppose that $x^* \neq y^*$. From Theorem 4.2, we have $\frac{F(E(x^*))}{G(E(x^*))} \prec_{LU} \frac{F(E(y^*))}{G(E(y^*))}$. For suitable $\lambda_k^L,\lambda_k^U \in R^+$, $k=1,\cdots,l$, we have

$$\sum_{k=1}^{l}\lambda_k^L\frac{f_k^L}{g_k^U}(E(x^*))+\sum_{k=1}^{l}\lambda_k^U\frac{f_k^U}{g_k^L}(E(x^*))<\sum_{k=1}^{l}\lambda_k^L\frac{f_k^L}{g_k^U}(E(y^*))+\sum_{k=1}^{l}\lambda_k^U\frac{f_k^U}{g_k^L}(E(y^*)). \tag{4.2}$$

Since $\frac{F}{G}$ is strict E-invex at y^*, and $h_j, j=1,\cdots,m$ are E-invex at y^*, using the similar proof method in Theorem 4.1, we obtain

$$\sum_{k=1}^{l}\lambda_k^L\frac{f_k^L}{g_k^U}(E(y^*))+\sum_{k=1}^{l}\lambda_k^U\frac{f_k^U}{g_k^L}(E(y^*))<\sum_{k=1}^{l}\lambda_k^L\frac{f_k^L}{g_k^U}(E(x^*))+\sum_{k=1}^{l}\lambda_k^U\frac{f_k^U}{g_k^L}(E(x^*)),$$

which contradicts (4.2). Therefore $x^*=y^*$, and the proof is completed. □

5. Conclusions

In this paper, under the *LU* order relationship, we developed the E-KKT optimality conditions for the optimization problem with a fractional interval objective function. We presented the idea of *LU*-*E*-invex for fractional interval valued functions, which generalize the *LU*-convexity for interval valued functions. We established the KKT conditions for the fractional interval problem under the *LU*-*E*-invex assumptions. Moreover, extending the idea of the gradient of the interval valued functions by using E-differential to the fractional interval valued functions, we derived the sufficient and necessary E-KKT conditions. In addition, we also established the E-dual theorem between primal problem and dual problems.

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