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WELL-POSEDNESS AND CONVERGENCE RESULTS FOR ELLIPTIC HEMIVARIATIONAL INEQUALITIES

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Abstract. We consider an elliptic hemivariational inequality in a real reflexive Banach space X which, under appropriate assumptions on the data, has a unique solution $u \in X$. We recall the concepts of well-posedness in the sense of Tykhonov and Levitin-Polyak for this inequality, and then we extend these concepts by introducing new well-posedness concepts, constructed with a larger set of approximating sequences. We also prove that, under additional assumptions, these new well-posedness concepts are optimal in the sense that all the sequences of elements of X which converge to the solution u are approximating sequences. This result, presented in Theorem 4.1, provides necessary and sufficient conditions for any sequence $\{u_n\} \subset X$ which guarantees that it converges to u and, therefore, it represents a convergence criterion to the solution of the hemivariational inequality. This criterion can be used in various applications. To provide an example, we illustrate its use in the study of a penalty method associated to an elliptic hemivariational inequality which describes the equilibrium of an elastic membrane in contact with a obstacle, the so-called foundation.

Keywords. Elliptic hamivariational inequality; Well-posedness results; Elastic membrane; Contact with normal compliance and unilateral constrain.
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1. INTRODUCTION

A large number of mathematical models in Physics, Mechanics, and Engineering Sciences are expressed in terms of nonlinear boundary value problems which lead, in their variational formulation, to various type of inequalities. Their analysis, including existence and uniqueness results, well-posedness results, optimal control, and error estimates for numerical schemes had made the object of many books and papers and, therefore, the literature in the field is extensive. Here we restrict ourselves to mention the books [9, 12, 15, 21, 22, 23] and, more recently, [1, 19, 25], as well as the survey paper [11]. The results presented in [1, 9, 22] concern the analysis of various classes of variational inequalities and are based on arguments of monotonicity and convexity, including properties of the subdifferential of a convex function. The results in [11, 12, 19, 21, 23, 25] concern the analysis of hemivariational inequalities and are based on properties

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of the subdifferential in the sense of Clarke, defined for locally Lipschitz functions, which may be nonconvex.

The concept of Tykhonov well-posedness was introduced in [29] for a minimization problem, and then it has been generalized for different optimization problems; see, e.g., [7, 13, 14, 16, 32]. It has been extended in the recent years to various mathematical problems like inequalities, inclusions, fixed point and saddle point problems. The well-posedness of variational inequalities was studied for the first time in [17, 18] and the study of well-posedness of hemivariational inequalities was initiated in [10]. References in the field include [30], among others. A general framework which unifies the well-posedness theories of abstract problems in metric spaces was recently introduced in [24]. There are, the concept of $\mathscr{T}$-well-posedness was introduced and studied, based on the notion of a new and nonstandard mathematical object, the so-called Tykhonov triple, denoted by $\mathscr{T}$. The two ingredients of the concept are the followings: a) any Tykhonov triple $\mathscr{T}$ generates a class of special sequences, the so-called $\mathscr{T}$-approximating sequences; b) a problem $\mathscr{P}$ is $\mathscr{T}$ well-posed if it has a unique solution and any $\mathscr{T}$-approximating sequence converge to this solution.

Given a problem $\mathscr{P}$, which has a unique solution u in a normed or metric space X, one of the main questions is to find a well-posedness concept such that any sequence which converge to the solution u is an approximating sequence. This question is strongly related to the problem of finding convergence criteria to the solution of the corresponding problem $\mathscr{P}$. Such kind of convergence criteria were obtained in [8, 26, 27] in the study of variational inequalities, minimization problems, fixed point problems, and differential equations, respectively. Moreover, they have been obtained in [28] the study of stationary inclusions.

In this current paper, we continue our research in [8] by considering the case elliptic hemivariational inequalities. The functional framework that we use everywhere in the rest of the paper is the following: X is a real reflexive Banach space, $K\subset X$, $A:X\to X^*$, $j:X\to\mathbb{R}$, and $f\in X^*$. Here and below, we use X^* for the dual of X, $\langle\cdot,\cdot\rangle$ for the duality paring between X^* and X and 0_X for the zero element of X. Moreover, we assume that the function j is locally Lipschitz and we use the notation $j^0(u;v)$ for the generalized directional derivative of j at $u\in X$, in the direction $v\in X$. With these notations, we consider the following inequality problem.

Problem $\mathscr{P}$. *Find an element u such that*

$$u\in K,\quad \langle Au,v-u\rangle+j^0(u;v-u)\geq\langle f,v-u\rangle\quad\forall\,v\in K. \tag{1.1}$$

Besides the mathematical interest in such kind of inequalities, our study is motivated by possible applications in Solid and Contact Mechanics. Indeed, a large number of mathematical models which describe the contact of an elastic body with an obstacle lead to variational formulations of the form (1.1) in which u represents the displacement field. References in the field are the books [19, 24, 25], for instance. Moreover, such an example of elastic contact problem will be provided in the last section of the current paper.

The rest of this paper is structured as follows. In Section 2, we introduce some preliminary material. Next, in Section 3, we recall the concepts of Tykhonov and Levitin-Polyak well-posedness for hemivariational inequality (1.1). We then state and prove a new convergence result, Theorem 3.1, and use it to extend the Tykhonov and Levitin-Polyak well-posedness concepts for inequality (1.1). In Section 4, we state and prove our main result, Theorem 4.1. It provides necessary and sufficient conditions, which guarantee the convergence of a sequence

$\{u_n\} \subset X$ to the solution u of Problem $\mathscr{P}$. Based on this result, we conclude that, under additional assumptions on the data, the new well-posedness concepts that we introduce in this paper are optimal. Finally, in Section 5, we apply these abstract results in the study of a mathematical model which describes an elastic membrane in contact with an obstacle, the so-called foundation.

2. Preliminaries

Everywhere in this paper, unless it is specified otherwise, we use the functional framework described in Introduction. Moreover, we denote by $\|\cdot\|_X$ the norm on X, and 2^X represents the set of parts of X. All the limits, lower limits and upper limits below are considered as $n \to \infty$ even if we do not mention it explicitly. The symbols "$\to$" and "$\rightharpoonup$" denote the strong and the weak convergence in various spaces which will be specified, except in the case that these convergence take place in $\mathbb{R}$. For a sequence $\{\varepsilon_n\} \subset \mathbb{R}_+$ which converges to zero, we use the short hand notation $0 \le \varepsilon_n \to 0$. Finally, we denote by $d(u,M)$ the distance between an element $u \in X$ and the set $M \subset X$, that is, $d(u,M) = \inf_{v\in M} \|u-v\|_X$.

Sets and Operators. For the subsets of X, we recall the following definition.

Definition 2.1. Let X be a normed space. A subset $K \subset X$ is called:

(a) *closed* if the limit of each convergent sequence of elements of K belongs to K, that is,

$$\{u_n\} \subset K, \quad u_n \to u \quad \text{in } X \implies u \in K.$$

(b) *weakly closed* if the limit of each weakly convergent sequence of elements of K belongs to K, that is,

$$\{u_n\} \subset K, \quad u_n \rightharpoonup u \quad \text{in } X \implies u \in K.$$

(c) *convex* if it has the property

$$u, v \in K \implies (1-t)u + t\,v \in K \qquad \forall t \in [0,1].$$

Evidently, every weakly closed subset of X is closed, but the converse is not true, in general. An exception is provided by the class of convex subsets of a Banach space which are strongly closed if and only are weakly closed.

We now recall the following definition for the convergence of sequences of subsets of X, introduced in [20].

Definition 2.2. Let X be a normed space, $\{K_n\}$ be a sequence of nonempty subsets of X, K be a nonempty subset of X. We say that the sequence $\{K_n\}$ converges to K *in the sense of Mosco* and we write $K_n \xrightarrow{M} K$ if the following conditions hold:

(a) for each $u \in K$, there exists a sequence $\{u_n\}$ such that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \to u$ in X.

(b) for each sequence $\{u_n\}$ such that $u_n \in K_n$ for each $n \in \mathbb{N}$ and $u_n \rightharpoonup u$ in X, we have $u \in K$.

Next, we recall the definition of some classes of operators.

Definition 2.3. Let X be a normed space. An operator $A\colon X \to X^*$ is called:

(a) *bounded* if A maps bounded sets of X into bounded sets of X^*.

(b) *monotone* if $\langle Au - Av, u - v\rangle \ge 0$ for all $u, v \in X$.

(c) *strongly monotone* if there exists $m_A > 0$ such that

$$\langle Au - Av, u - v\rangle \geq m\|u - v\|_X^2 \quad \forall\, u,\, v \in X.$$

(d) *pseudomonotone* if it is bounded and $u_n \rightharpoonup u$ in X together with

$$\limsup\, \langle Au_n, u_n - u\rangle \leq 0$$

imply that

$$\liminf\, \langle Au_n, u_n - v\rangle \geq \langle Au, u - v\rangle \quad \forall\, v \in X.$$

(e) *hemicontinuous* if, for all u, v, $w \in X$, the function $\lambda \mapsto \langle A(u + \lambda v), w\rangle$ is continuous on $[0,1]$.

(f) *Lipschitz continuous* if there exists $L_A > 0$ such that

$$\|Au - Av\|_X \leq L_A\|u - v\|_X \quad \forall\, u,\, v \in X.$$

Examples and various properties of the nonlinear operators which satisfy the definitions above can be found in [4, 5, 19, 31], for instance. For the following result, we refer the reader to [6, Section 1.9] and [31, Proposition 27.6].

Proposition 2.1. *Let A, $B\colon X \to X^*$ be operators on a reflexive Banach space X. The following statements hold:*

(a) *if A is bounded, hemicontinuous and monotone, then it is pseudomonotone;*

(b) *if A and B are pseudomonotone, then $A + B$ is pseudomonotone.*

Locally Lipschitz functions. We now recall the basic definitions and properties of the generalized subdifferential in the sense of Clarke [2].

Definition 2.4. Let X be a Banach space. A function $j\colon X \to \mathbb{R}$ is said to be *locally Lipschitz*, if, for every $x \in X$, there exist U_x a neighborhood of x and a constant $L_x > 0$ such that $|j(y) - j(z)| \leq L_x\|y - z\|_X$ for all y, $z \in U_x$.

We note that a convex continuous function $j\colon X \to \mathbb{R}$ is locally Lipschitz. Also, if a function $j\colon X \to \mathbb{R}$ is Lipschitz continuous on bounded sets of X, then it is locally Lipschitz, while the converse does not hold, in general.

Definition 2.5. Let $j\colon X \to \mathbb{R}$ be a locally Lipschitz function. The *generalized (Clarke) directional derivative* of j at $x \in X$ in the direction $v \in X$, denoted by $j^0(x; v)$, is defined by

$$j^0(x; v) = \limsup_{y \to x,\ \lambda \downarrow 0} \frac{j(y + \lambda v) - j(y)}{\lambda}.$$

The *subdifferential in the sense of Clarke* (or, equivalently, the *generalized gradient*) of j at x, denoted by $\partial j(x)$, is a subset of the dual space X^* given by

$$\partial j(x) = \Big\{\, \zeta \in X^* \ :\ j^0(x; v) \geq \langle \zeta, v\rangle \quad \forall\, v \in X \,\Big\}.$$

A locally Lipschitz function j is said to be *regular* (in the sense of Clarke) at $x \in X$ if for all $v \in X$ the one-sided directional derivative

$$j'(x; v) = \lim_{\lambda \downarrow 0} \frac{j(x + \lambda v) - j(x)}{\lambda}$$

exists and $j^0(x; v) = j'(x; v)$.

The following result collects some properties of the generalized directional derivative and the generalized gradient.

Proposition 2.2. *Assume that $j\colon X \to \mathbb{R}$ is a locally Lipschitz function on a Banach space X. Then, the following hold.*

(i) *For every $x \in X$, the function $X \ni v \mapsto j^0(x;v) \in \mathbb{R}$ is positively homogeneous, i.e., $j^0(x;\lambda v) = \lambda j^0(x;v)$ for all $\lambda \geq 0$ and subadditive, i.e., $j^0(x;v_1+v_2) \leq j^0(x;v_1) + j^0(x;v_2)$ for all v_1, $v_2 \in X$.*

(ii) *The function $X \times X \ni (x,v) \mapsto j^0(x;v) \in \mathbb{R}$ is upper semicontinuous, i.e., for all x, $v \in X$, $\{x_n\}$, $\{v_n\} \subset X$ such that $x_n \to x$ and $v_n \to v$ in X, $\limsup j^0(x_n;v_n) \leq j^0(x;v)$.*

(iii) *For every x, $v \in X$, we have $j^0(x;v) = \max \Big\{ \langle \zeta, v\rangle \,:\, \zeta \in \partial j(x) \Big\}$.*

We refer to [3, 5, 19] for additional results on the generalized gradient, its relation to classical notions of differentiability, and other calculus rules.

An existence and uniqueness result. In the study of (1.1), we consider the following assumptions

$$K \text{ is a nonempty closed convex subset of } X. \tag{2.1}$$

$$\begin{cases} A\colon X \to X^* \text{ is pseudomonotone and strongly monotone, i.e.:} \\ \text{(a) } A \text{ is bounded and } u_n \rightharpoonup u \text{ in } X \text{ with } \limsup \langle Au_n, u_n - u\rangle \leq 0 \\ \qquad \text{implies that } \liminf \langle Au_n, u_n - v\rangle \geq \langle Au, u-v\rangle\ \forall\, v \in X. \\ \text{(b) there exists } m_A > 0 \text{ such that} \\ \qquad \langle Au - Av, u - v\rangle \geq m_A \|u-v\|_X^2 \quad \forall\, u,\, v \in X. \end{cases} \tag{2.2}$$

$$\begin{cases} j\colon X \to \mathbb{R} \text{ is such that:} \\ \text{(a) } j \text{ is locally Lipschitz.} \\ \text{(b) } \|\xi\|_{X^*} \leq c_0 + c_1 \|v\|_X \ \ \forall\, v \in X,\ \xi \in \partial j(v) \\ \qquad \text{with } c_0, c_1 \geq 0. \\ \text{(c) there exists } \alpha_j \geq 0 \text{ such that} \\ \qquad j^0(v_1; v_2 - v_1) + j^0(v_2; v_1 - v_2) \leq \alpha_j \|v_1 - v_2\|_X^2 \\ \qquad \forall\, v_1, v_2 \in X. \end{cases} \tag{2.3}$$

$$\alpha_j < m_A. \tag{2.4}$$

$$f \in X^*. \tag{2.5}$$

The unique solvability of inequality (1.1) is given by the following result.

Theorem 2.1. *Assume (2.1)–(2.5). Then, inequality (1.1) has a unique solution $u \in K$.*

A proof of Theorem 2.1 can be found in [25], based on a surjectivity argument for pseudomonotone multivalued operators.

3. Well Posedness Results

We start this section by recalling the concept of Tykhonov well-posedness and Levitin well-posedness for hemivariational inequality (1.1). For more details and results in the field, we refer to [24, 30].

Definition 3.1. A sequence $\{u_n\} \subset X$ is called an approximting sequence for the hemivariational inequality (1.1) if there exists $0 \le \varepsilon_n \to 0$ such that

$$u_n \in K, \quad \langle Au_n, v - u_n\rangle + j^0(u_n; v-u_n) + \varepsilon_n \|v - u_n\|_X \ge \langle f, v-u_n\rangle \quad \forall v \in K,\ n \in \mathbb{N}.$$

Problem $\mathscr{P}$ is well-posed in the sense of Tykhonov if it has a unique solution u and every approximating sequence converges in X to u.

Definition 3.2. A sequence $\{u_n\} \subset X$ is called an *LP*-approximting sequence for the hemivariational inequality (1.1) if there exists two sequences $\{w_n\} \subset X$ and $\{\varepsilon_n\} \subset \mathbb{R}$ such that $w_n \to 0_X$ in X, $0 \le \varepsilon_n \to 0$ and

$$u_n + w_n \in K, \quad \langle Au_n, v - u_n\rangle + j^0(u_n; v-u_n) + \varepsilon_n \|v - u_n\|_X \ge \langle f, v-u_n\rangle \quad \forall v \in K,\ n \in \mathbb{N}.$$

Problem $\mathscr{P}$ is well-posed in the sense of Levitin-Polyak if it has a unique solution u and every *LP*-approximating sequence converges in X to u.

It is easy to see that any approximating sequence is an *LP*-approximating sequence. Therefore, if problem $\mathscr{P}$ is well-posed in the sense of Levitin-Polyak, then it is well-posed in the sense of Tykhonov, too. Some elementary examples (see [26], for instance) can be constructed in order to see that the converse of this statement is not true. We conclude from here that the Levitin-Polyak well-posedness of inequality (1.1) implies its Tykhonov well-posedness and, for this reason, we say that the Levitin-Polyak well-posedness concept is "stronger" than the Tykhonov well-posedness concept. With this remark in mind, our aim in this section is to introduce three new well-posedness concepts in the study of inequality (1.1) which are stronger (in the sense defined above) than the Levitin-Polyak and Tykhonov well-posedness concepts. Our construction requires some preliminaries that we introduce in what follows. First, everywhere below we assume that (2.1)–(2.5) hold, and we denote by u the solution of Problem $\mathscr{P}$ provided by Theorem 2.1. Moreover, we assume the following additional condition on the function j

$$\left\{\begin{array}{l} \text{For all sequence } \{u_n\} \subset X \text{ such that} \\ u_n \rightharpoonup u \text{ in } X \text{ and for any } v \in X, \text{ we have} \\ \limsup j^0(u_n; v-u_n) \le j^0(u; v-u). \end{array}\right. \tag{3.1}$$

Note that this condition can be avoided in the proof of Theorem 3.1 below. Nevertheless, we keep it for two reasons: first, it allows us to simplify the proof of this theorem; second, it is satisfied in the example we present in Section 5 below. Moreover, we mention that examples of functions j which satisfy this conditions are given in [25].

Next, given a sequence $\{u_n\} \subset X$, we consider the following statements:

(S_1) $\quad d(u_n, K) \to 0,$

(S_2) $\quad \exists \{w_n\} \subset X$ such that $w_n \to 0_X$ in X and $u_n + w_n \in K \;\; \forall n \in \mathbb{N},$

(S_3) $\quad \exists \{K_n\} \subset 2^X$ such that $K_n \xrightarrow{M} K$ and $u_n \in K_n \;\; \forall n \in \mathbb{N},$

(S_4) $\quad \exists\{\varepsilon_n\}\to 0$ such that

$$\langle Au_n, v-u_n\rangle + j^0(u_n; v-u_n) + \varepsilon_n(1+\|v-u_n\|_X) \geq \langle f, v-u_n\rangle \quad \forall v\in K,\ n\in\mathbb{N}.$$

(S_5) $\quad u_n \to u$ in X.

In addition, we define the following sets of sequences:

$$\mathscr{T}_1 = \Big\{ \{u_n\}\subset X \ : \{u_n\} \text{ satisfies the statements } (S_1) \text{ and } (S_4) \Big\},$$

$$\mathscr{T}_2 = \Big\{ \{u_n\}\subset X \ : \{u_n\} \text{ satisfies the statements } (S_2) \text{ and } (S_4) \Big\},$$

$$\mathscr{T}_3 = \Big\{ \{u_n\}\subset X \ : \{u_n\} \text{ satisfies the statements } (S_3) \text{ and } (S_4) \Big\},$$

$$\mathscr{T}_{\mathscr{P}} = \Big\{ \{u_n\}\subset X \ : \{u_n\} \text{ satisfies the statements } (S_5) \Big\}.$$

The inclusions between these sets represent the main result in this section and is provided by the following theorem.

Theorem 3.1. *Assume* (2.1)–(2.5). *Then:*

a) $\mathscr{T}_1 = \mathscr{T}_2 \subset \mathscr{T}_3$.

b) $\mathscr{T}_3 \subset \mathscr{T}_{\mathscr{P}}$ *if, in addition,* (3.1) *holds.*

Everywhere below we assume that the hypothesis of Theorem 2.1 hold, even if we do not mention it explicitly. The proof of Theorem 3.1 is based on some preliminary results that we state and prove in what follows.

Lemma 3.1. *The statements* (S_1) *and* (S_2) *are equivalent.*

Proof. We recall that any reflexive Banach space X can be always considered as equivalently renormed strictly convex. Therefore, without loosing the generality, we assume that $(X, \|\cdot\|_X)$ is strictly convex and, using assumption (2.1), we can consider the projection operator on $P_K : X \to K$, defined by

$$\xi = P_K f \quad \Longleftrightarrow \quad \xi\in K \quad \text{and} \quad \|\xi - f\|_X = d(f,K) \qquad \forall f\in K. \tag{3.2}$$

Assume that (S_1) holds, i.e., $d(u_n,K)\to 0$ and, for each $n\in\mathbb{N}$, take $w_n = P_K u_n - u_n$. Then, it is easy to see from (3.2) that $u_n + w_n \in K$ for each $n\in\mathbb{N}$ and, moreover,

$$\|w_n\|_X = \|u_n - P_K u_n\|_X = d(u_n,K)\to 0.$$

This shows that $\{w_n\}$ satisfies the requirements in (S_2).

Conversely, if (S_2) holds, then $d(u_n,K)\leq \|u_n-(u_n+w_n)\|_X = \|w_n\|_X$ for each $n\in\mathbb{N}$. Since $w_n\to 0_X$, we deduce that (S_1) holds, which concludes the proof. □

Lemma 3.2. *Statement* (S_2) *implies statement* (S_3).

Proof. Let $\{u_n\}\subset X$ be a sequence which satisfies condition (S_2), and let $\{w_n\}\subset X$ be a sequence such that $w_n\to 0_X$ in X and $u_n+w_n\in K$, for each $n\in\mathbb{N}$. For each $n\in\mathbb{N}$, we define the set K_n by equality $K_n = K + B(0_X, \|w_n\|_X)$, where $B(0_X,\|w_n\|_X)$ represents the ball of radius $\|w_n\|_X$ centered in 0_X, i.e.,

$$B(0_X,\|w_n\|_X) = \Big\{ v\in X \ : \ \|v\|_X \leq \|w_n\|_X \Big\}.$$

We claim that the following property holds:

$$K_n \xrightarrow{M} K. \tag{3.3}$$

Indeed, since $K \subset K_n$ for each $n \in \mathbb{N}$, it follows that the condition (a) in Definition 2.2 is satisfied. Next, let $v \in K$ and let $\{v_n\} \subset X$ be a sequence such that $v_n \in K_n$ for each $n \in \mathbb{N}$ and $v_n \rightharpoonup v$ in X. Then, for each $n \in \mathbb{N}$, there exist some elements $\widetilde{v}_n$ and $\widetilde{w}_n$ such that

$$v_n = \widetilde{v}_n + \widetilde{w}_n, \quad \widetilde{v}_n \in K, \quad \widetilde{w}_n \in B(0_X, \|w_n\|_X),$$

which means that $\|\widetilde{w}_n\|_X \leq \|w_n\|_X$. Thus assumption (S_2) implies that $\widetilde{w}_n \to 0_X$ in X. We combine this convergence with the convergence $v_n \rightharpoonup v$ in X and equality $v_n = \widetilde{v}_n + \widetilde{w}_n$ to see that $\widetilde{v}_n \rightharpoonup v$ in X. Thus, since K is weakly closed, we deduce that $v \in K$. This proves that the condition (b) in Definition 2.2 is satisfied, which concludes the proof of (3.3).

Finally, we write $u_n = (u_n + w_n) - w_n$ to see that $u_n \in K_n$ for all $n \in \mathbb{N}$. We now use this inclusion and implication (3.3) to see that the sequence $\{K_n\}$ satisfies the requirements in the convergence (S_3). □

Lemma 3.3. *Any sequence $\{u_n\}$ which satisfies the statement (S_4) is bounded.*

Proof. Let $v_0 \in K$ be fixed. Using assumption (2.2)(b), we have

$$m_A \|v_0 - u_n\|_X^2 \leq \langle Av_0 - Au_n, v_0 - u_n \rangle = \langle Av_0, v_0 - u_n \rangle - \langle Au_n, v_0 - u_n \rangle$$

and, therefore, inequality (3.2) yields

$$m_A \|v_0 - u_n\|_X^2 \leq \|Av_0 - f\|_{X^*} \|v_0 - u_n\|_X + j^0(u_n; v_0 - u_n) + \varepsilon_n (1 + \|v_0 - u_n\|_X).$$

Moreover, using assumption (2.3) and the properties of the Clarke directional derivative, we have

$$\begin{aligned} j^0(u_n; v_0 - u_n) &= j^0(u_n; v_0 - u_n) + j^0(v_0; u_n - v) - j^0(v_0; u_n - v_0) \\ &\leq j^0(u_n; v_0 - u_n) + j^0(v_0; u_n - v_0) + |j^0(v_0; u_n - v_0)| \\ &\leq \alpha_j \|u_n - v_0\|_X^2 + \Big| \max \{ \langle \xi, u_n - v_0 \rangle : \xi \in \partial j(v_0) \} \Big| \\ &\leq \alpha_j \|u_n - v_0\|_X^2 + (c_0 + c_1 \|v_0\|_X) \|u_n - v_0\|_X. \end{aligned}$$

Thus

$$j^0(u_n; v_0 - u_n) \leq \alpha_j \|u_n - v_0\|_X^2 + (c_0 + c_1 \|v_0\|_X) \|u_n - v_0\|_X, \tag{3.4}$$

which implies that

$$\begin{aligned} (m_A - \alpha_j) \|u_n - v_0\|_X^2 &\leq \|Av - f\|_{X^*} \|u_n - v_0\|_X + \varepsilon_n + \varepsilon_n \|u_n - v_0\|_X \\ &\quad + (c_0 + c_1 \|v_0\|_X) \|u_n - v_0\|_X. \end{aligned}$$

Using the smallness assumption (2.4), we deduce that there exists two positive constants C and D, which depend on v_0 and the rest of the data but do not depend on n, such that

$$\|u_n - v_0\|_X^2 \leq C(1 + \varepsilon_n) \|u_n - v_0\|_X + D\varepsilon_n.$$

We now use the elementary inequality

$$x^2 \leq ax + b \quad \Longrightarrow \quad x \leq a + \sqrt{b} \qquad \forall\, x,\, a,\, b \geq 0.$$

to see that

$$\|u_n - v_0\|_X \leq C(1+\varepsilon_n) + \sqrt{D\varepsilon_n}.$$

Since $0 \leq \varepsilon_n \to 0$, we deduce that $\{u_n - v_0\}$ is bounded in X. This implies that $\{u_n\}$ is a bounded sequence in X, which concludes the proof of the lemma. □

We now have all the ingredients to provide the proof of the theorem.

Proof of Theorem 3.1. a) This part of the theorem represents a direct consequence of Lemmas 3.1 and 3.2, combined with the definitions of the sets $\mathscr{T}_1$, $\mathscr{T}_2$, and $\mathscr{T}_3$.

b) We now move to the proof of the second part of the theorem. To this end, we assume that (3.1) holds and we consider a sequence $\{u_n\} \in \mathscr{T}_3$. Then, there exists a sequence $\{K_n\} \subset 2^X$ and a sequence $n \in \mathbb{N}$ such that

$$K_n \xrightarrow{M} K, \tag{3.5}$$

$$u_n \in K_n \quad \forall n \in \mathbb{N}, \tag{3.6}$$

$$\varepsilon_n \to 0, \tag{3.7}$$

and the inequality in (S_4) holds. We next prove that $u_n \to u$ in X. To this end, we divide the proof in two steps, described below.

Step i) $\{u_n\}$ *converges weakly to the solution* u *of Problem* $\mathscr{P}$. Using Lemma 3.3 and the reflexivity of the space X we deduce that, passing to a subsequence if necessary,

$$u_n \rightharpoonup \widetilde{u} \quad \text{in} \quad X, \quad \text{as} \quad n \to \infty, \tag{3.8}$$

with some $\widetilde{u} \in X$. Then, using assumptions (3.5), (3.6) and Definition 2.2, it follows that

$$\widetilde{u} \in K. \tag{3.9}$$

Consider now an arbitrary element $v \in K$. We use the inequality in (S_4) to see that

$$\langle Au_n, u_n - v\rangle \leq j^0(u_n; v-u_n) + \varepsilon_n(1+\|v-u_n\|_X) + \langle f, u_n - v\rangle. \tag{3.10}$$

Moreover, the convergence (3.7), (3.8), and assumption (3.1) imply that

$$\limsup j^0(u_n; v-u_n) \leq j^0(\widetilde{u}; v-\widetilde{u}), \tag{3.11}$$

$$\varepsilon_n(1+\|v-u_n\|_X) \to 0, \tag{3.12}$$

$$\langle f, u_n - v\rangle \to \langle f, \widetilde{u} - v\rangle. \tag{3.13}$$

We now pass to the upper limit in inequality (3.10) and use (3.11)–(3.13) to find that

$$\limsup \langle Au_n, u_n - v\rangle \leq j^0(\widetilde{u}; v-\widetilde{u}) + \langle f, \widetilde{u} - v\rangle. \tag{3.14}$$

Recall that this inequality holds for each $v \in K$. Next, we take $v = \widetilde{u}$ in (3.14) and use the property $j^0(\widetilde{u}; 0_X) = 0$ of the Clarke directional derivative to deduce that

$$\limsup \langle Au_n, u_n - \widetilde{u}\rangle \leq 0. \tag{3.15}$$

Exploiting now the pseudomonotonicity of the operator A, from (3.8) and (3.15), we have

$$\langle A\widetilde{u}, \widetilde{u} - v\rangle \leq \liminf \langle Au_n, u_n - \widetilde{u}\rangle \quad \forall v \in X. \tag{3.16}$$

Next, from (3.9), (3.16), and (3.14) we obtain that $\widetilde{u}$ is a solution to Problem $\mathscr{P}$, as claimed. Thus, by the uniqueness of the solution we find that $\widetilde{u} = u$.

A careful analysis of the results presented above indicates that every subsequence of $\{u_n\}$ which converges weakly in X has the same weak limit u. On the other hand, $\{u_n\}$ is bounded in X. Therefore, using a standard argument we deduce that the whole sequence $\{u_n\}$ converges weakly to u in X, as $n\to\infty$, which concludes the proof of this step.

Step ii) $\{u_n\}$ converges strongly to the solution of Problem $\mathscr{P}$. We now take $v=u=\widetilde{u}$ in inequalities (3.16) and (3.15) to see that $0\le \liminf\langle Au_n,u_n-u\rangle\le \limsup\langle Au_n,u_n-u\rangle\le 0$, which shows that

$$\langle Au_n,u_n-u\rangle\to 0. \tag{3.17}$$

On the other hand, the convergence (3.8) combined with equality $\widetilde{u}=u$ yields

$$\langle Au,u_n-u\rangle\to 0. \tag{3.18}$$

We now use the strong monotonicity of the operator A and write

$$m_A\|u_n-u\|_X^2\le\langle Au_n,u_n-u\rangle-\langle Au,u_n-u\rangle,$$

and then we use the convergence (3.17) and (3.18) to se that $u_n\to u$ in X.

We proved above that any $\mathscr{T}_3$-approximating sequence converges to the solution of Problem $\mathscr{P}$. Therefore, $\mathscr{T}_3\subset\mathscr{T}_{\mathscr{P}}$, which concludes the proof. □

Next, inspired by Definitions 3.1 and 3.2 now introduce the following definitions for $i\in\{1,2,3\}$.

Definition 3.3. A sequence $\{u_n\}\subset X$ is called a $\mathscr{T}_i$-approximting sequence for hemivariational inequality (1.1) if $\{u_n\}\in\mathscr{T}_i$. Problem $\mathscr{P}$ is $\mathscr{T}_i$-well-posed if it has a unique solution u and every $\mathscr{T}_i$-approximating sequence converges to u in X, that is, $\mathscr{T}_i\subset\mathscr{T}_{\mathscr{P}}$.

Remark 3.1. From Definition 3.3 and Theorem 3.1 we deduce the following.

a) Problem $\mathscr{P}$ is $\mathscr{T}_1$-well-posed if and only if it is $\mathscr{T}_2$-well-posed.

b) If Problem $\mathscr{P}$ is $\mathscr{T}_3$-well-posed; then it is $\mathscr{T}_1$- and $\mathscr{T}_2$-well-posed, too.

c) If conditions (2.1)–(2.5) and (3.1) are satisfied, then Problem $\mathscr{P}$ is $\mathscr{T}_i$-well-posedness, for any $i\in\{1,2,3\}$.

d) If Problem $\mathscr{P}$ is $\mathscr{T}_i$-well-posed with some $i\in\{1,2,3\}$, then it is Tyknonov and Levitin-Polyak well-posed, too.

e) If conditions (2.1)–(2.5) and (3.1) are satisfied, then Problem $\mathscr{P}$ is Tyknonov and Levitin-Polyak well-posed.

Note that the statements c) and e) in Remark 3.1 allows us to identify classes of sequences $\{u_n\}\subset X$ which converge to the solution u. For instance, e) shows that, under assumptions (2.1)–(2.5) and (3.1), any approximating sequence as well as any *LP*-approximating sequences converges to u in X. This result is useful in various applications since, for instance, it can be used to prove the continuous dependence of the solution with respect to f. Indeed, if $u=u(f)$ denotes the solution of inequality (1.1) with $f\in X^*$ and $u_n=u(f_n)$ is the solution of inequality (1.1) with $f_n\in X^*$, it is easy to see that u_n satisfies (3.1) with $\varepsilon_n=\|f_n-f\|_{X^*}$ for any $n\in\mathbb{N}$. Therefore, if $f_n\to f$ in X^*, $\{u_n\}$ is an approximating sequence, $u_n\to u$ in X, we conclude that

$$f_n\to f\ \text{ in }\ X^*\quad\Longrightarrow\quad u(f_n)\to u(f)\ \text{ in }\ X,$$

which represents a continuous dependence result, as claimed.

We now proceed with the following example.

Example 3.1. Consider inequality (1.1) in the particular case $X=\mathbb{R}$, $K=[0,1]$ $Au=u$, $j\equiv 0$, and $f=2$. The solution of this problem is $u=P_K f=1$. Let $\{u_n\}\subset\mathbb{R}$ be the sequence given by $u_n=1-\frac{1}{n}$ for all $n\in\mathbb{N}$. Then $u_n\to u$. Taking $w_n=\varepsilon_n=\frac{1}{n}$ for any $n\in\mathbb{N}$, it is easy to see that $\{u_n\}$ is a $\mathscr{T}_2$-approximating sequence. Nevertheless, $\{u_n\}$ is not a *LP*-approximating sequence. Indeed, assume that $\{u_n\}$ is a *LP*-approximating sequence. Then, there exists $0\le\varepsilon_n\to 0$ such that, for all $n\in\mathbb{N}$, $u_n(v-u_n)+\varepsilon_n|v-u_n|\ge f(v-u_n)$ for all $v\in[0,1]$. We now fix $n\in\mathbb{N}$, take $v=1-\frac{1}{2n}$ in the previous inequality, and use equalities $u_n=1-\frac{1}{n}$, $f=2$ to deduce that $\varepsilon_n\ge 1+\frac{1}{n}$, which contradicts the convergence $\varepsilon_n\to 0$.

Example 3.1 below shows that there exist sequences $\{u_n\}\subset X$ which converge to u but are neither approximating sequences, nor *LP*- approximating sequences. We conclude from here that the neither the Tykhonov nor the Levitin-Polyak well-posedness of Problem $\mathscr{P}$ (see Remark 3.1 e)) can be invoked to prove the convergence of these sequences. In contrast, the $\mathscr{T}_i$-well posedness result with $i\in\{1,2,3\}$ (see Remark 3.1 c)) can be used to deduce the convergence of the corresponding sequence to $u=1$. This proves that any $\mathscr{T}_i$-well-posedness result implies Levitin-Polyak and Tykhonov well-posedness results. For this reason, as explained in the previous section, we say that the $\mathscr{T}_i$-well-posedness concepts are stronger than the classical Tykhonov and Levitin-Polyak well-posedness concepts introduced above, in the study of inequality (1.1).

We end this section by recalling that Theorem 3.1 proves the inclusion

$$\mathscr{T}_3\subset\mathscr{T}_{\mathscr{P}}. \tag{3.19}$$

The question if this inclusion is strict deserves to be studied. This question is left open under assumption (2.1)–(2.5) and (3.1). Nevertheless, we shall provide an answer to this question in the next section, under additional assumptions on the operator A and function j.

4. A Convergence Criterion

In this section, we state and prove a convergence criterion for the solution of inclusion (1.1). To this end, we assume (2.1)–(2.5) and (3.1) even if we do not mention it explicitly, and we denote by u the solution of the hemivariational inequality (1.1) provided by Theorem 2.1. We also consider the following additional assumptions

$$A: X\to X^* \text{ is a continuous operator.} \tag{4.1}$$

$$\left\{\begin{array}{l}\text{There exists a function } c_j:\mathbb{R}_+\to\mathbb{R}_+ \text{ such that}\\ j^0(u,v_1)-j^0(u,v_2)\le c_j(\|u\|_X)\|v_1-v_2\|_X \quad \text{for all } u,v_1,v_2\in X.\end{array}\right. \tag{4.2}$$

$$\left\{\begin{array}{l}\text{There exists a continuous function } d_j:\mathbb{R}_+\to\mathbb{R}_+ \text{ such that } d(0)=0 \text{ and}\\ j^0(u_1;v)-j^0(u_2;v)\le d_j(\|u_1-u_2\|_X)\|v\|_X \quad \text{for all } u_1,u_2,v\in X.\end{array}\right. \tag{4.3}$$

Our main result in this section is the following.

Theorem 4.1. *Assume (2.1)-(2.5), (3.1) and, in addition, assume (4.1)-(4.3). Then* $\mathscr{T}_1=\mathscr{T}_2=\mathscr{T}_3=\mathscr{T}_{\mathscr{P}}$.

Proof. Let $\{u_n\} \subset \mathcal{T}_{\mathscr{P}}$, which implies that $u_n \to u$ in X. We fix $n \in \mathbb{N}$ and $v \in K$. Since $u \in K$, it follows that

$$d(u_n, u) \leq \|u_n - u\|_X \to 0. \tag{4.4}$$

Next, we write

$$\begin{aligned}&\langle Au_n, v - u_n\rangle + j^0(u_n, v - u_n) - \langle f, v - u_n\rangle\\ &\quad = \langle Au_n - Au, v - u_n\rangle + \langle Au, v - u\rangle + \langle Au, u - u_n\rangle\\ &\qquad + j^0(u_n; v - u_n) - j^0(u; v - u) + j^0(u; v - u) - \langle f, v - u\rangle + \langle f, u_n - u\rangle.\end{aligned}$$

Using (1.1), we deduce that

$$\begin{aligned}&\langle Au_n, v - u_n)_X + j^0(u_n; v - u_n) - (f, v - u_n)_X\\ &\quad \geq \langle Au_n - Au, v - u_n\rangle + \langle Au, u - u_n\rangle + j^0(u_n; v - u_n) - j^0(u; v - u) + \langle f, u_n - u\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}&\langle Au_n, v - u_n\rangle + j^0(u_n; v - u_n)\\ &\quad + \langle Au_n - Au, u_n - v\rangle + \langle Au, u_n - u\rangle + j^0(u; v - u) - j^0(u_n; v - u_n) + \langle f, u - u_n\rangle \geq \langle f, v - u_n\rangle.\end{aligned} \tag{4.5}$$

We now use standard arguments to see that

$$\langle Au_n - Au, u_n - v\rangle \leq \|Au_n - Au\|_{X^*}\|u_n - v\|_X, \tag{4.6}$$

$$\langle Au, u_n - u\rangle \leq \|Au\|_{X^*}\|u_n - u\|_X, \tag{4.7}$$

$$\langle f, u_n - u\rangle \leq \|f\|_{X^*}\|u_n - u\|_X. \tag{4.8}$$

Moreover, we write

$$\begin{aligned}&j^0(u; v - u) - j^0(u_n; v - u_n)\\ &\quad = \Big[j^0(u; v - u) - j^0(u; v - u_n)\Big] + \Big[j^0(u; v - u_n) - j^0(u_n; v - u_n)\Big],\end{aligned}$$

and then we use assumptions (4.2) and (4.3) to see that

$$j^0(u; v - u) - j^0(u_n; v - u_n) \leq c_j(\|u\|_X)\|u_n - u\|_X + d_j(\|u_n - u\|)_X\|v - u_n\|_X. \tag{4.9}$$

We now combine inequalities (4.5)–(4.9) to see that

$$\begin{aligned}&\langle Au_n, v - u_n\rangle + j^0(u_n, v - u_n) + \big[\|Au_n - Au\|_{X^*} + d_j(\|u_n - u\|)\big]\|u_n - v\|_X\\ &\qquad + \big[\|Au\|_{X^*} + c_j(\|u\|_X) + \|f\|_{X^*}\big]\|u_n - u\|_X \geq \langle f, v - u_n\rangle.\end{aligned} \tag{4.10}$$

Then, using notation

$$\varepsilon_n = \max\Big\{\|Au_n - Au\|_{X^*} + d_j(\|u_n - u\|), \big[\|Au\|_{X^*} + c_j(\|u\|_X) + \|f\|_{X^*}\big]\|u_n - u\|_X\Big\} \tag{4.11}$$

and inequality (4.10), we see that the inequality in (3.2) holds. Moreover, assumptions (4.1), (4.3), and notation (4.11) imply that $\varepsilon_n \to 0$.

It follows from above that condition (S_4) is satisfied. Therefore, since the convergence (4.4) shows that (S_1) holds, too, we deduce that $\{u_n\}$ is a $\mathcal{T}_1$-approximating sequence, i.e., $\{u_n\} \in \mathcal{T}_1$. To conclude we proved that $\mathcal{T}_{\mathscr{P}} \subset \mathcal{T}_1$. Recall also that Theorem 3.1 guarantees that

$\mathscr{T}_1 = \mathscr{T}_2 \subset \mathscr{T}_3 \subset \mathscr{T}_{\mathscr{P}}$. Therefore, we deduce that $\mathscr{T}_1 = \mathscr{T}_2 = \mathscr{T}_3 = \mathscr{T}_{\mathscr{P}}$, which concludes the proof. □

We now complete the statement of Theorem 4.1 with the following comments.

First, we remark that Theorem 4.1 provides necessary and sufficient conditions which guarantee the convergence of a sequence $\{u_n\} \in X$ to the solution of hemivariational inequality (1.1). Therefore, it represents a *convergence criterion* to the solution of Problem $\mathscr{P}$. On the other hand, it provides an answer to the question that we stated at the end of Section 3, since it shows that under assumptions the additional assumptions (4.1)–(4.3) the inclusion (3.19) is not strict.

Next, we recall that, under the assumptions of this theorem, any convergent sequence is a $\mathscr{T}_i$-approximating sequence with $i \in \{1,2,3\}$ and conversely. This shows that, under these assumptions, the $\mathscr{T}_i$-well-posedness concepts introduced in Definition 3.3 are optimal, since the sets of $\mathscr{T}_i$-approximating sequences are such that $\mathscr{T}_i = \mathscr{T}_{\mathscr{P}}$, that is, they are equal to the largest subset of the set $\mathscr{T}_{\mathscr{P}}$.

Finally, recall that Theorem 4.1 was proved under the additional assumptions (4.1)–(4.3) on the operator A and function j. Relaxing or removing these assumptions represent an important issue which deserves to be studied in the future.

5. An Example

In this section, we provide an application of Theorems 2.1 and 3.1 in the analysis of a non-smooth nonlinear boundary value problems with unilateral constraints. The problem can be stated as follow.

Problem $\mathscr{M}$. Find $u : \Omega \to \mathbb{R}$ and $\xi : \Omega \to \mathbb{R}$ such that

$$u \le g, \quad \mu\Delta u + \xi + f_0 \ge 0, \quad (u-g)(\mu\Delta u + \xi + f_0) = 0 \quad \text{in} \quad \Omega, \tag{5.1}$$

$$-\xi \in \partial q(u) \quad \text{in} \quad \Omega, \tag{5.2}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{5.3}$$

Here $\Omega \subset \mathbb{R}^2$ is a regular domain with boundary Γ, g and μ are positive constants, and f_0 and q are given functions. This problem models the equilibrium of an elastic membrane which occupies the domain Ω, is fixed on its boundary and is in contact along its surface with an obstacle, the so-called foundation. The unknown u is the vertical displacement of the membrane, μ is the Lamé coefficient and f_0 represents the density of applied body force. The obstacle is assumed to be made of a rigid body covered of a layer of deformable material with thickness g. The unknown ξ represents the reaction of this layer. The model (5.1)–(5.3) is obtained by taking into account the equilibrium equation, the normal compliance contact condition for the deformable layer and the Signorini contact condition for the rigid body. It represents a two-dimensional version of various models of contact studied in [24, 25], for instance.

In parallel with problem $\mathscr{M}$, we consider a second model of contact which can be formulated as follows.

Problem $\mathscr{M}_\lambda$. Find $u_\lambda : \Omega \to \mathbb{R}$, $\xi_\lambda : \Omega \to \mathbb{R}$, and $\eta_\lambda : \Omega \to \mathbb{R}$ such that

$$\left.\begin{array}{l} u_\lambda \le \widetilde{g},\ \ \mu\Delta u_\lambda + \xi_\lambda + \eta_\lambda + f_0 \ge 0, \\ (u_\lambda - \widetilde{g})(\mu\Delta u_\lambda + \xi_\lambda + \eta_\lambda + f_0) = 0 \end{array}\right\} \quad \text{in} \quad \Omega, \tag{5.4}$$

$$-\xi_\lambda \in \partial q(u) \quad \text{in} \quad \Omega, \tag{5.5}$$

$$-\eta_\lambda = \frac{1}{\lambda}\, p(u_\lambda - g) \quad \text{in} \quad \Omega, \tag{5.6}$$

$$u_\lambda = 0 \quad \text{on} \quad \Gamma, \tag{5.7}$$

where $\widetilde{g} \ge g$ and $\lambda > 0$ are given constants. Problem $\mathscr{M}_\lambda$ describes a similar physical setting. Nevertheless, here we assume that the obstacle is made by a rigid body covered with two layers: the first one of thickness $\widetilde{g} - g$ and the second one of thickness g. We model the reaction of these two layers with the term $\xi_\lambda + \eta_\lambda$, in which ξ_λ and η_λ satisfy the normal compliance conditions (5.5) and (5.6), respecively. There, λ represents the deformability coefficient of the first layer and, therefore, $\frac{1}{\lambda}$ is its stiffness coefficient. Moreover, p is a normal compliance function which will be described below.

We now turn to the variational formulation of Problem $\mathscr{M}$ and $\mathscr{M}_\lambda$. To this end, we use the short hand notation V for the Sobolev space $H_0^1(\Omega)$ endowed with the inner product

$$(u, v)_V = (\nabla u, \nabla v)_{L^2(\Omega)^2} \qquad \forall\, u, v \in V \tag{5.8}$$

and the associated norm $\|\cdot\|_V$. Recall that the Friedrichs-Poincaré inequality guarantee that V is a Hilbert space. We denote in what follows by V^* the dual of V and by $\langle\cdot,\cdot\rangle$ the duality pairing between V^* and V. Moreover, 0_V represents the zero element of V. In addition, we recall that the inclusion $V \subset L^2(\Omega)$ is compact and there exists a constant $c_0 > 0$, which depend only on Ω such that

$$\|u\|_{L^2(\Omega)} \le c_0 \|u\|_V \qquad \forall\, u \in V. \tag{5.9}$$

Next, we consider the following assumptions on the data

$$\left\{\begin{array}{l} q\colon \Omega \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \text{(a) } q(\cdot, r) \text{ is measurable on } \Omega \text{ for all } r \in \mathbb{R} \text{ and there} \\ \qquad \text{exists } \overline{e} \in L^2(\Omega) \text{ such that } q(\cdot, \overline{e}(\cdot)) \in L^1(\Omega), \\ \text{(b) } q(x, \cdot) \text{ is locally Lipschitz on } \mathbb{R} \text{ for a.e. } x \in \Omega, \\ \text{(c) } |\partial q(x, r)| \le \overline{c}_0 + \overline{c}_1\, |r| \text{ for a.e. } x \in \Gamma_3, \text{ for all } r \in \mathbb{R} \text{ with } \overline{c}_0, \overline{c}_1 \ge 0, \\ \text{(d) } q^0(x, r_1; r_2 - r_1) + q^0(x, r_2; r_1 - r_2) \le L_q\, |r_1 - r_2|^2 \\ \qquad \text{for a.e. } x \in \Omega, \text{ all } r_1, r_2 \in \mathbb{R} \text{ with } L_q \ge 0. \\ \text{(e) either } q(x, \cdot) \text{ or } -q(x, \cdot) \text{ is regular on } \mathbb{R}, \text{ for a.e. } x \in \Omega. \end{array}\right. \tag{5.10}$$

$$\left\{\begin{array}{l} p\colon \Omega \times \mathbb{R} \to \mathbb{R}_+ \text{ is such that} \\ \text{(a) there exists } L_p > 0 \text{ such that} |p(x, r_1) - p(x, r_2)| \le L_p |r_1 - r_2| \\ \qquad\quad \text{for all} \, r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega, \\ \text{(b) } (p(x, r_1) - p(x, r_2))\, (r_1 - r_2) \ge 0 \\ \qquad\quad \text{for all} \, r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega, \\ \text{(c) } p(\cdot, r) \text{ is measurable on } \Omega \text{ for all } r \in \mathbb{R}, \\ \text{(d) } p(x, r) = 0 \text{ if and only if } r \le 0, \text{ a.e. } x \in \Omega. \end{array}\right. \tag{5.11}$$

$$c_0^2 L_q < \mu. \tag{5.12}$$

$$f_0 \in L^2(\Omega). \tag{5.13}$$

Moreover, we recall the inequalities

$$\widetilde{g} \geq g > 0. \tag{5.14}$$

Next, we define the sets K and $\widetilde{K}$, the operators $A\colon V \to V^*$ and $G\colon V \to V^*$, the function $j : V \to \mathbb{R}$ and the element $f \in V^*$ by equalities

$$K = \{\, v \in V : v \leq g \ \text{ a.e. in } \Omega\,\}, \tag{5.15}$$

$$\widetilde{K} = \{\, v \in V : v \leq \widetilde{g} \ \text{ a.e. in } \Omega\,\}, \tag{5.16}$$

$$\langle Au, v\rangle = \mu \int_\Omega \nabla u \cdot \nabla v \, dx \qquad \forall\, u,\, v \in V, \tag{5.17}$$

$$\langle Gu, v\rangle = \int_\Omega p(u-g)\, v\, dx \qquad \forall\, u,\, v \in V, \tag{5.18}$$

$$j(v) = \int_\Omega q(v)\, dx \qquad \forall\, v \in V, \tag{5.19}$$

$$\langle f, v\rangle_V = \int_\Omega f_0 v\, dx \qquad \forall\, v \in V. \tag{5.20}$$

With these preliminaries, we are in a position to derive the variational formulation of Problem $\mathscr{M}$ and $\mathscr{M}_\lambda$. Assume that (u,ξ) is a regular solution to Problem $\mathscr{M}$ which implies that

$$u \in K. \tag{5.21}$$

Consider an arbitrary element $v \in K$. We write

$$\mu\Delta u\,(v-u) = (\mu\Delta u + \xi + f_0)(v-g) + (\mu\Delta u + \xi + f_0)(g-u) - \xi\,(v-u) - f_0(v-u)$$

a.e. in Ω, then we use (5.1) and the definition (5.15) of the set K to find that

$$\mu\Delta u\,(v-u) \leq -\xi\,(v-u) - f_0(v-u) \qquad \text{a.e. in } \Omega. \tag{5.22}$$

Next, recall that inclusion (5.2) and the definition of the directional derivative implies that

$$-\,\xi\,(v-u) \leq q^0(u; v-u) \qquad \text{a.e. in } \Omega. \tag{5.23}$$

We now combine inequalities (5.22) and (5.23), and then integrate the resulting inequality on Ω to see that

$$\int_\Omega \mu\Delta u\,(v-u)\, dx \leq \int_\Omega q^0(u; v-u)\, dx - \int_\Omega f_0(v-u)\, dx. \tag{5.24}$$

Finally, we recall the identity

$$\int_\Omega \Delta u\, v\, dx = -\int_\Omega \nabla u \cdot \nabla v\, dx, \tag{5.25}$$

valid for any $u,\, v \in V$. Combining relations (5.24) and (5.25), we obtain that

$$\mu \int_\Omega \nabla u \cdot (\nabla v - \nabla u)\, dx + \int_\Omega q^0(u; v-u)\, dx \geq \int_\Omega f_0(v-u)\, dx. \tag{5.26}$$

On the other hand, a standard argument based on assumption (5.10) on the function q and definition (5.19) of the function j show that the function $j : V \to \mathbb{R}$ is locally Lipschitz, satisfy conditions (2.3) on the space $X = V$ and, moreover,

$$j^0(u;v) = \int_\Omega q^0(u;v)\,dx \qquad \forall\, u,\ v \in V. \tag{5.27}$$

We now use (5.21), (5.26), (5.27), and the definitions (5.17), (5.20) to deduce the following variational formulation of Problem $\mathscr{M}$, in terms of displacement.

Problem $\mathscr{M}^V$. *Find a displacement field u such that the inequality below holds:*

$$u \in K, \qquad \langle Au, v-u\rangle + j^0(u;v-u) \geq \langle f, v-u\rangle \quad \forall\, v \in K. \tag{5.28}$$

Similar arguments, based on the definitions (5.15) and (5.18) of set $\widetilde{K}$ and operator G, lead to the following variational formulation of contact problem $\mathscr{M}_\lambda$.

Problem $\mathscr{M}^V_\lambda$. *Find a displacement field u_λ such that the inequality below holds:*

$$u_\lambda \in \widetilde{K}, \quad \langle Au_\lambda, v-u_\lambda\rangle + \frac{1}{\lambda}\langle Gu_\lambda, v-u_\lambda\rangle + j^0(u_\lambda; v-u_\lambda) \geq \langle f, v-u\rangle \quad \forall\, v \in \widetilde{K}. \tag{5.29}$$

Our main result in this section is the following.

Theorem 5.1. *Assume (5.10)–(5.14). Then Problem $\mathscr{M}^V$ has a unique solution $u \in K$, and for each $\lambda > 0$, Problem $\mathscr{M}^V_\lambda$ has a unique solution $u_\lambda \in \widetilde{K}$. Moreover, $u_\lambda \to u$ in V as $\lambda \to 0$.*

Proof. The proof is divided into four steps, as follows.

Step i). Unique solvability of Problem $\mathscr{M}^V$. We use Theorem 2.1 on $X = V$. To this end, we use definition (5.15) to see that condition (2.1) is satisfied. In addition, using (5.8), it is easy to see that linear operator (5.17) satisfies condition (2.2) with $m_A = L_A = \mu$. Moreover, as already mentioned, function (5.19) satisfies condition (2.3). To compute α_j, we use equality (5.27), assumption (5.10)(d), and inequality (5.9) to see that

$$j^0(u;v-u) + j^0(v;u-v) = \int_\Omega \left[q^0(u;v-u) + q^0(v;u-v)\right] dx$$

$$\leq L_q \int_\Omega |u-v|^2\,dx \leq L_q c_0^2 \|u-v\|_V^2 \qquad \forall\, u, v \in V.$$

It follows from here that the corresponding constant α_j in (2.3)(d) is $\alpha_j = c_0^2 L_q$. Then, using equality $m_A = \mu$ and (5.12), we find that the smallness condition (2.4) holds, too. Finally, recall that the element f in (5.20) satisfies condition (2.5). The existence of a unique solution to Problem $\mathscr{M}^V$ is now a direct consequence of Theorem 2.1.

Step ii). Some properties of the operator G. In this step, we state and prove some properties of the operator G, which are used in the rest of the proof. Thus we claim that the operator G defined by (5.18) has the following properties.

$$\left\{\begin{array}{ll} \text{(a)} & G : V \to V^* \ \text{ is a Lipschitz continuous monotone operator.} \\ \text{(b)} & \langle Gu, v-u\rangle \leq 0 \qquad \forall\, u \in \widetilde{K},\ v \in K. \\ \text{(c)} & u \in \widetilde{K}, \quad \langle Gu, v-u\rangle = 0 \quad \forall\, v \in K \implies u \in K. \end{array}\right. \tag{5.30}$$

To prove this claim, we consider three elements u, v, and w in V. We use definition (5.18), assumption (5.11)(a), and inequality (5.9) to see that

$$|\langle Gu - Gv, w\rangle| = \Big|\int_\Omega \big[p(u-g) - p(v-g)\big]\, w\, dx\Big|$$

$$\leq \int_\Omega |p(u-g) - p(v-g)|\,|w|\, dx \leq L_p \int_\Omega |u-v|\,|w|\, dx \leq L_p c_0^2 \|u-v\|_V \|w\|_V.$$

This inequality implies that $\|Gu - Gv\|_{V^*} \leq L_p c_0^2 \|u-v\|_V$ and shows that G is Lipschitz continuous with constant $L_G = L_p c_0^2$. On the other hand, using again (5.18) and (5.11)(b), we deduce that

$$\langle Gu - Gv, u - v\rangle = \int_\Omega \big[p(u-g) - p(v-g)\big]\big[(u-g) - (v-g)\big]\, dx \geq 0,$$

which shows that G is a monotone operator. We conclude from above that (5.30)(a) holds.

Assume now that $u \in \widetilde{K}$ and $v \in K$. We use assumption (5.11)(d) to see that $p(v-g) = 0$ a.e. in Ω. Thus

$$\langle Gu, v - u\rangle = \int_\Omega p(u-g)(v-u)\, dx = \int_\Omega \big[p(u-g) - p(v-g)\big]\big[(v-g) - (u-g)\big]\, dx.$$

Now, it follows from assumption (5.11)(b) that $\langle Gu, v-u\rangle \leq 0$, which shows that condition (5.30)(b) holds.

Assume now that $u \in \widetilde{K}$ and $\langle Gu, v - u\rangle = 0$ for all $v \in K$, which implies that

$$\int_\Omega p(u-g)(v-g)\, dx = \int_\Omega p(u-g)(u-g)\, dx \quad \forall\, v \in K. \tag{5.31}$$

Recall that (5.11)(b) and (d) guarantee that

$$p(u-g)(v-g) \leq 0 \quad \text{a.e. in } \Omega, \quad \forall\, v \in K, \tag{5.32}$$

$$p(u-g)(u-g) \geq 0 \quad \text{a.e. in } \Omega. \tag{5.33}$$

We now combine equality (5.31) with inequalities (5.32) and (5.33) to find that

$$\int_\Omega p(u-g)(u-g)\, dx = 0. \tag{5.34}$$

Next, (5.33) and (5.34) imply that $p(u-g)(u-g) = 0$ a.e. in Ω. Using condition (5.11) (d) again, we find that $u \leq g$ a.e. in Ω. This shows that (5.30)(c) holds and concludes the proof of this step.

Step iii). Unique solvability of Problem $\mathscr{M}_\lambda^V$. We let $\lambda > 0$ and use (5.30) (a) to deduce that the operator $G : V \to V^*$ defined by (5.18) it is bounded, hemicontinuous, and monotone. Then, using Proposition 2.1 (a), it follows that it is pseudomonotone and monotone. Moreover, assumption (2.2) combined with inequality $\lambda > 0$ and Proposition 2.1 (b) shows that the operator

$$A + \frac{1}{\lambda} G : V \to V^*$$

is pseudomonotone and strongly monotone with constant $m_{A+\frac{1}{\lambda}G} = \mu$, that is, it satisfies condition (2.2). Recalling assumption (5.16), we see that we are in a position to use Theorem 2.1 again to deduce the unique solvability of Problem $\mathscr{M}_\lambda^V$.

Step iv). Proof of the convergence $u_\lambda \to u$ as $\lambda \to 0$. We shall use Theorem 3.1 (b) on space V. To this end, we claim that condition (3.1) is satisfied. Indeed, if $u_n \rightharpoonup u \in V$ and $v \in V$, using a standard compactness argument, identity (5.27), and Proposition (2.2)(ii) we have

$$\limsup j^0(u_n; v-u_n) = \limsup \int_\Omega q^0(u_n; v-u_n)\,dx \tag{5.35}$$

$$\leq \int_\Omega \limsup q^0(u_n; v-u_n)\,dx \leq \int_\Omega q^0(u; v-u)\,dx = j^0(u; v-u), \tag{5.36}$$

which proves that condition (3.1) is satisfied.

Consider now a sequence $\{\lambda_n\}$ such that

$$\lambda_n > 0 \quad \forall\, n \in \mathbb{N}, \quad \lambda_n \to 0 \quad \text{as } n \to \infty \tag{5.37}$$

and, for each $n \in \mathbb{N}$, denote by u_n the solution of inequality (5.29) for $\lambda = \lambda_n$. We have

$$u_n \in \widetilde{K}, \quad \langle Au_n, v-u_n\rangle + \frac{1}{\lambda_n}\langle Gu_n, v-u_n\rangle + j^0(u_n; v-u_n) \geq \langle f, v-u_n\rangle \quad \forall\, v \in \widetilde{K}. \tag{5.38}$$

We use the property (5.30)(a) of operator G to see that $\langle Gu_n, v-u_n\rangle \leq 0$ for all $v \in K$, $n \in \mathbb{N}$ and, therefore, (5.38) implies that

$$\langle Au_n, v-u_n\rangle + j^0(u_n; v-u_n) \geq \langle f, v-u_n\rangle \quad \forall\, v \in K,\ n \in \mathbb{N}. \tag{5.39}$$

This shows that

$$\text{The sequence } \{u_n\} \text{ satisfies condition } (S_4). \tag{5.40}$$

We now use Lemma 3.3, the reflexivity of V, and a standard compactness argument to see that by passing to a subsequence, if necessary, we have

$$u_n \rightharpoonup \widetilde{u} \ \text{ in } \ V, \ \text{ as } \ n \to \infty, \tag{5.41}$$

with some $\widetilde{u} \in V$. Moreover, inclusion $u_n \in \widetilde{K}$ for each $n \in \mathbb{N}$, the properties of set (5.16) and the convergence (5.41) imply that $\widetilde{u} \in \widetilde{K}$. Let v be an element in $\widetilde{K}$ and let $n \in \mathbb{N}$. Then, (5.38) implies that

$$\frac{1}{\lambda_n}\langle Gu_n, u_n - v\rangle \leq \langle Au_n, v-u_n\rangle + j^0(u_n, v-u_n) + \langle f, u_n - v\rangle. \tag{5.42}$$

Next, using the arguments similar to those used in the proof of (3.4) and the boundedness of $\{u_n\}$, we deduce that each term in the right hand side of inequality (5.42) is bounded. This implies that there exists a constant $M_0 > 0$, which depends on v but does not depend on n such that $\langle Gu_n, u_n - v\rangle \leq \lambda_n M_0$. We now pass to the upper limit in this inequality and use the convergence (5.37) to deduce that

$$\limsup \langle Gu_n, u_n - v\rangle \leq 0. \tag{5.43}$$

We now take $v = \widetilde{u} \in \widetilde{K}$ in (5.43) and find that

$$\limsup \langle Gu_n, u_n - \widetilde{u}\rangle \leq 0. \tag{5.44}$$

Therefore, using the pseudomonotonicity of operator G, guaranteed by property (5.30)(a) and Proposition 2.1 (a), we obtain that

$$\liminf \langle Gu_n, u_n - v\rangle \geq \langle G\widetilde{u}, \widetilde{u} - v\rangle. \tag{5.45}$$

We now combine inequalities (5.45) and (5.43) to find that $\langle G\widetilde{u}, \widetilde{u} - v\rangle \leq 0$, and recall that this inequality holds for any $v \in \widetilde{K}$. On the other hand, by (5.30)(b), we deduce that $\langle G\widetilde{u}, \widetilde{u} - v\rangle \geq 0$

for any $v \in K$. Now, since (5.14) implies that $K \subset \widetilde{K}$, we obtain from above that $\langle G\widetilde{u}, \widetilde{u} - v\rangle = 0$ for all $v \in K$. We now use the property (5.30)(c) of operator G to deduce that $\widetilde{u} \in K$.

Next, we use (5.38) and inclusion $K \subset \widetilde{K}$ again, to obtain that

$$\langle Au_n, u_n - v\rangle \leq \frac{1}{\lambda_n}\langle Gu_n, v - u_n\rangle + j^0(u_n, v - u_n) + \langle f, u_n - v\rangle,$$

v being an arbitrary element of K. Therefore, using (5.30)(b), we find that

$$\langle Au_n, u_n - v\rangle \leq j^0(u_n, v - u_n) + \langle f, u_n - v\rangle. \tag{5.46}$$

We now pass to the upper limit in this inequality, and use (5.41) and the property (3.1) of the function j to deduce that

$$\limsup \langle Au_n, u_n - v\rangle \leq j^0(\widetilde{u}, v - \widetilde{u}) + \langle f, \widetilde{u} - v\rangle. \tag{5.47}$$

Now, taking $v = \widetilde{u} \in K$ in (5.47) and using Proposition 2.2 (a), we obtain that

$$\limsup \langle Au_n, u_n - \widetilde{u}\rangle \leq 0. \tag{5.48}$$

On the other hand, using the strong monotonicity of operator A, we have

$$m_A\|u_n - \widetilde{u}\|_V^2 \leq \langle Au_n, u_n - \widetilde{u}\rangle - \langle A\widetilde{u}, u_n - \widetilde{u}\rangle$$

and, using (5.41) and (5.48), we deduce that

$$u_n \to \widetilde{u} \quad \text{in } V, \tag{5.49}$$

which shows that $d(u_n, K) \leq \|u_n - \widetilde{u}\|_V \to 0$, i.e., condition (S_1) holds. A careful analysis demonstrates that this property is valid for any weakly convergent subsequence of sequence $\{u_n\}$. To conclude, it follows from above that

$$\left\{\begin{array}{l}\text{Any weakly convergent subsequence of the sequence } \{u_n\} \\ \text{satisfies condition } (S_1).\end{array}\right. \tag{5.50}$$

We now combine (5.40) and (5.50) to deduce that any weakly convergent subsequence of the sequence $\{u_n\}$ satisfies conditions (S_1) and (S_4), i.e., it belongs to the set $\mathscr{T}_1$. Using now Theorem 3.1 it follows that

$$\left\{\begin{array}{l}\text{Any weakly convergent subsequence of the sequence } \{u_n\} \\ \text{converges in } V \text{ to the solution } u \text{ of inequality } (1.1).\end{array}\right. \tag{5.51}$$

Finally, we argue by contradiction and, to this end, we assume that the convergence $u_n \to u$ in V does not hold. Then, there exists $\delta_0 > 0$ such that for all $k \in \mathbb{N}$ there exists $u_{n_k} \in X$ such that

$$\|u_{n_k} - u\|_X \geq \delta_0. \tag{5.52}$$

Note that $\{u_{n_k}\}$ is a subsequence of sequence $\{u_n\}$ and, therefore, Step i) and Lemma 3.3 imply that it is bounded in X. We now employ the reflexivity of V to deduce that there exists a subsequence of sequence $\{u_{n_k}\}$, again denoted by $\{u_{n_k}\}$, which is weakly convergent in X. Then, (5.51) guarantees that $u_{n_k} \to u$ as $k \to \infty$. We now pass to the limit when $k \to \infty$ in (5.52) and find that $\delta_0 \leq 0$. This contradicts inequality $\delta_0 > 0$. Therefore, $u_n \to u$ in V.

To summarize, we proved that for any sequence $\{\lambda_n\}$ which satisfies condition (5.37) we have that $u_{\lambda_n} \to u$ in V, which concludes the proof of the theorem. □

In addition to the mathematical interest in this convergence, it is important from mechanical point of view since it shows that the weak solution of Problem $\mathscr{M}$ can be approached by the week solution of Problem $\mathscr{M}_\lambda$ for a large stiffness coefficient.

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