Appl. Set-Valued Anal. Optim. 7 (2025), No. 2, pp. 223-249
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.7.2025.2.06

AN INERTIAL SCHEME FOR APPROXIMATING THE SOLUTIONS OF PSEUDO-MONOTONE EQUILIBRIUM AND COMMON FIXED POINT PROBLEMS IN BANACH SPACES

OLUWATOSIN TEMITOPE MEWOMO*, EMEKA CHIGAMEZU GODWIN, TIMILEHIN OPEYEMI ALAKOYA

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa

Abstract. In this paper, we propose a new iterative method for approximating a common solution of pseudo-monotone equilibrium problems and common fixed point problems of Bregman quasi-nonexpansive mappings in a p-uniformly convex and uniformly smooth Banach space. Under some weaker conditions, we obtain a strong convergence result. A variational inequality problem is also considered. We present numerical examples to illustrate and support our main convergence theorem.
Keywords. Halpern iteration; Inertial method; Pseudo-monotone equilibium; Variational inequalities.
2020 Mathematics Subject Classification. 65K15, 90C30.

1. INTRODUCTION

Let C be a nonempty, convex, and closed set in a real Banach space E whose dual is denoted by E^*. The Equilibrium Problem (EP) for a bifuntion $f : C \times C \to \mathbb{R}$ consists of finding a point x^* in set C such that

$$f(x^*, x) \geq 0, \ \forall\, x \in C. \tag{1.1}$$

We borrow EP(f) to denote the solution set of problem (1.1), which was inaugurated as the Ky Fan inequality (see [9]). It is noteworthy to mention that numerous problems in economics, transportation, and so on can be modelled into the equilibrium problem, which include many hot problems, such as, variational ineqaulity problems, inclusion problems, saddle point problems and so on as special cases; see, e.g., [10, 14, 26, 28, 36]. Recently, various iterations were investigated for solutions of the equilibrium problem; see, e.g., [5, 6, 22, 37, 38] and the references therein.

In [16], Korpelevich proposed the extragradient method (EM) for a monotone variational inequality, which a variant of the equilibrium problem, in the framework of Euclidean spaces as follows:

$$\begin{cases} y_n = P_C(x_n - \rho A(x_n)), \\ x_{n+1} = P_C(x_n - \rho A(y_n)), \end{cases}$$

*Corresponding author.
E-mail address: mewomoo@ukzn.ac.za (O.T. Mewomo).
Received 11 January 2024; Accepted 10 March 2025; Published online 25 March 2025.

where A is K-Lipschitz continuous and monotone and $\rho \in (0, \frac{1}{K})$. Notice that the EM needs to evaluate two projections (onto feasible set C) in each iteration, which is a major drawback that might affect the applications of the EM. For various modified EM, we refer to [1, 7, 12].

Inspired by the EM, Quoc, Muu, and Nguyen [27] in the framework of finite dimensional Hilbert spaces studied the following iterative scheme for approximating a solution of pseudo-monotone EP:

$$\begin{cases} x_0 \in C \subset \mathbb{R}^N, \\ y_n = \arg\min\{\rho f(x_n, y) + \frac{1}{2}\|y - x_n\|^2 : \ y \in C\} \\ x_{n+1} = \arg\min\{\rho f(y_n, y) + \frac{1}{2}\|y - x_n\|^2 \ : y \in C\}, \ n \geq 0, \end{cases} \tag{1.2}$$

where $0 < \rho < \min\{\frac{1}{2k_1}, \frac{1}{2k_2}\}$, and k_1 and k_2 are Lipschitz constants of the pseudo-monotone bifunction f. They proved that the iterative sequence generated by (1.2) converges to a solution of (1.1). We note that Algorithm (1.2) requires solving two convex optimization programs over feasible set C. This requirement is considered a strong assumption as it affects the efficiency and applicability of the method in the case that feasible set C has a complex structure.

The celegrated subgradient extragradient method (SEM) is one of the methods proposed recently to overcome this limitation. The SEM involves the replacement of the second projection in the EM with a projection onto a simple, constructible half-space, which has an explicit formula for computation.

In [12], Hieu extended the SEM to solve the equilibrium problem in the framework of infinite dimensional real Hilbert spaces as follows:

$$\begin{cases} y_n = \arg\min\{\rho_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 \ : y \in C\} \\ T_n = \{y \in H : \langle x_n - \rho_n w_n - y_n, y - y_n\rangle \leq 0\} \\ z_n = \arg\min\{\rho_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 \ : y \in T_n\} \\ x_{n+1} = \gamma_n x_0 + (1 - \gamma_n) z_n, \end{cases} \tag{1.3}$$

where $w_n \in \partial_2 f(x_n, y_n)$, $0 < \rho_n < \min\{\frac{1}{2k_1}, \frac{1}{2k_2}\}$, with k_1, k_2 being the Lipschitz constants of the bifunction. Hieu proved a strong convergent result in real Hilbert spaces for approximating a solution of the pseudo-monotone equilibrium problem by using the Halpern subgradient extragradient method (1.3). We note that the Lipschitz constants k_1, k_2 of the bifunction f are often unknown or difficult to estimate, which could be a major barrier in the implementation the algorithm.

Recently, Muangchoo [21] studied a viscosity-type scheme for solving pseudo-monotone equilibrium problems with a step-size technique which requires prior knowledge of the Lipschitz constants of the bifunction. The proposed method is as follows:

$$\begin{cases} y_n = \arg\min\{\chi_n f(x_n, y) + \frac{1}{2}\|x_n - y\|^2 \ : y \in C\} \\ \varepsilon_n = \{z \in H : \langle x_n - \chi_n w_n - y_n, z - y_n\rangle \leq 0\} \\ z_n = \arg\min\{\mu\chi_n f(y_n, y) + \frac{1}{2}\|x_n - y\|^2 \ : y \in \varepsilon_n\} \\ x_{n+1} = \gamma_n g(x_n) + (1 - \gamma_n) z_n, \end{cases}$$

where $w_n \in \partial_2 f(x_n, y_n)$, $0 < \sigma < \min\{1, \frac{1}{2k_1}, \frac{1}{2k_2}\}$, $\mu \in (0, \sigma)$. Muangchoo a proved strong convergence result. However, there exists few methods in the literature for solving EP without relying on the prior knowledge of the Lipschitz constants for the bifunction f.

The Fixed Point Problem (shortly FPP) finds plenty applications in numerous real-world problems. Let $F(D)$ denote the fixed point set of D, i.e., $F(D) := \{\hat{x} \in C : \hat{x} = D\hat{x}\}$, where $D : C \to C$ is a nonlinear mapping. Inertial methods were extensively investigated for the acceleration of various numerical schemes. A discrete counterpart of a second order dissipative dynamical system in time best describes this approach. The approach was initially in [24] for roughly solving issues involving smooth convex minimization. In [2], Beck and Teboulle improved Nesterov's acceleration for the situation of structured convex minimization. For recent inertial technique, we refer to [8, 11, 18–20, 32, 34] and the references therein.

In this paper, we propose and analyze a Halpern-type subgradient extragradient algorithm, which is more general than related works done in real Hilbert spaces, for approximating a common solution of the pseudo-monotone equilibrium problem and the fixed point problem of a finite family of Bregman quasi-nonexpansive mappings in the context of p-uniformly convex Banach spaces. In order to improve the convergence rate of our algorithm, we also include the inertial factor. Additionally, by choosing an appropriate self-adaptive step size that is independent of any prior knowledge of the Lipschitz constants of the pseudo-monotone bifunction, we are able to achieve a strong convergence theorem and our result finds associated applications in solving the classical variational inequality problem. In addition, the examination of the strong convergence is succinct and does not rely on the typical "Two-cases Approach" that is frequently employed in many works to ensure strong convergence. (see, e.g., [7, 12]). The finding in this paper mainly expands and generalizes the findings in [12, 21, 40].

The remainder of this paper is structured as follows: We review several concepts and existing lemmas needed for the sequel in Section 2. We give our suggested method and highlight a few of its highlights in Section 3. The strong convergence analysis is investigated in Section 4. And the variational inequality is also considered in Section 4. We present several numerical experiments in Section 5 to evaluate the effectiveness of our method. Section 6 presents a concluding remark.

2. Preliminaries

Strong and weak convergence are indicated in the following by "$\to$" and "$\rightharpoonup$", respectively. By $w_\omega(x_n)$, we represent the collection of all weak cluster points of $\{x_n\}$. Assume that the unit sphere of E is represented by $U(E) := \{\hat{x} \in E : \|\hat{x}\| = 1\}$. The function $\beta_E : (0,2] \to [0,1]$ represents the modulus of convexity denoted by

$$\beta_E(\varepsilon) = \inf\Big\{1 - \frac{\|\hat{x}+\hat{y}\|}{2} : \hat{x},\hat{y} \in U(E),\ \|\hat{x}-\hat{y}\| \geq \varepsilon\Big\}.$$

Recall that space E is said to be uniformly convex if $\beta_E(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$. E is said to be p-uniformly convex if and only if $c_p > 0$ such that $\beta_E(\varepsilon) \geq c_p\varepsilon^p$ for all $\varepsilon \in (0,2]$. It should be noted that all p-uniformly convex space is uniformly convex. In what follows, the function $\nabla_E : \mathbb{R}^+ := [0,\infty) \to \mathbb{R}^+$ is the modulus of E described by

$$\nabla_E(\tau) = \sup\Big\{\frac{\|\hat{x}+\tau\hat{y}\| + \|\hat{x}-\tau\hat{y}\|}{2} - 1 : \hat{x},\hat{y} \in U(E)\Big\}.$$

The space E is known to be uniformly smooth if $\frac{\nabla_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. In the case $q > 1$, a Banach space E is said to be q-uniformly smooth if $z_q > 0$ such that $\rho_E(\tau) \leq z_q\tau^q$ for all $\tau > 0$.

A Banach space E is known to be p-uniformly convex if and only if its dual E^* is q-uniformly smooth, and p and q fulfill $\frac{1}{p}+\frac{1}{q}=1$ (see [41]).

If $g : E \to (-\infty,+\infty]$ is a proper, lower semicontinuous, and convex function, then the Frenchel conjugate of g denoted by $g^* : E^* \to (-\infty,+\infty]$ is described by $g^*(v) = \sup\{\langle \hat{x}, v\rangle - g(\hat{x}) : \hat{x} \in E\}$ for all $v \in E^*$. Let the domain of g be denoted by $(domg) = \{\hat{x} \in E : g(\hat{x}) < +\infty\}$. Then, for any $\hat{x} \in int(domg)$ and $\hat{y} \in E$, the right-hand derivative of g at $\hat{x}$ in the direction $\hat{y}$ is defined by $g^0(\hat{x},\hat{y}) = \lim_{t\to 0^+} \frac{g(\hat{x}+t\hat{y})-g(\hat{x})}{t}$. It is claimed that g is Gâteaux differentiable at $\hat{x}$ if $\lim_{t\to 0^+} \frac{g(\hat{x}+t\hat{y})-g(\hat{x})}{t}$ exists for any $\hat{y}$. More precisely, $g^0(\hat{x},\hat{y})$ intercepts with $\bigtriangledown g(\hat{x})$ (the value of the gradient $\bigtriangledown g$ of g at $\hat{x}$)). If g is Gâteaux differentiable for any $\hat{x} \in int(domg)$, it is said to be Gâteaux differentiable. g is said to be Fréchet differentiable at $\hat{x}$, if its limit is uniformly reached in $\|\hat{y}\| = 1$ whereas g is said to be uniformly Fréchet differentiable on a subset C of E, if the above limit is uniformly reached for $\hat{x} \in C$ and $\|\hat{y}\| = 1$.

A function g is said to be Legendre if it satisfies the following conditions:

(1) The interior of the domain of g, $int(domg)$ is nonempty, g is Gâteaux differentiable on $int(domg)$ and $dom \bigtriangledown g = int(domg)$.

(2) The interior of the domain of g^*, $int(domg^*)$ is nonempty, g^* is Gâteaux differentiable on $int(domg^*)$ and $dom \bigtriangledown g^* = int(domg^*)$.

Definition 2.1. [3] Let $g : E \to (-\infty,+\infty]$ be a convex and Gâteaux differentiable function. The function $\Delta_g : E \times E \to [0,+\infty)$ defined by $\Delta_g(\hat{x},\hat{y}) := g(\hat{x}) - g(\hat{y}) - \langle \bigtriangledown g(\hat{y}), \hat{x}-\hat{y}\rangle$ is called the Bregman distance with respect to g.

The Bregman distance possess the following interesting properties:

(i) $\Delta_g(\hat{x},\hat{x}) = 0$, but $\Delta_g(\hat{x},\hat{y}) = 0$ does not necessarily imply that $\hat{x} = \hat{y}$,

(ii) for $\hat{x} \in domg$ and $\hat{y},\hat{z} \in int(domg)$, $\Delta_g(\hat{x},\hat{y}) + \Delta_g(\hat{y},\hat{z}) - \Delta_g(\hat{x},\hat{z}) \leq \langle \bigtriangledown g(\hat{z}) - \bigtriangledown g(\hat{y}), \hat{x}-\hat{y}\rangle$,

(iii) for each $\hat{z} \in E$, $\{x_i\}_{i=1}^N$ and $\{\alpha_n\}_{i=1}^N \subset (0,1)$ with $\sum_{i=1}^N \alpha_i = 1$, such that

$$\Delta_g\left(\hat{z}, \nabla g^*\left(\sum_{i=1}^N \alpha_i \nabla g(x_i)\right)\right) \leq \sum_{i=1}^N \alpha_i \Delta_g(\hat{z}, x_i).$$

It is a known fact that the Bregman distance Δ_g is not a metric because it fails to satisfy the symmetric and triangular inequality properties. Moreover, it is well known that the duality mapping J_p^E is the sub-differential of the functional $g_p(.) = \frac{1}{p}\|.\|^p$ for $p > 1$; see [4]. The Bregman distance Δ_p is defined with respect to g_p as follows:

$$\begin{aligned}\Delta_p(x,y) = \frac{1}{p}\|x\|^p - \frac{1}{p}\|y\|^p - \langle J_E^p y, x-y\rangle &= \frac{1}{p}\|x\|^p - \langle J_E^p y, x\rangle + \frac{1}{q}\|y\|^p \\ &= \frac{1}{q}\|y\|^p - \frac{1}{q}\|x\|^p - \langle J_E^p y - J_E^p x, x\rangle.\end{aligned}$$

Definition 2.2. Let C be a nonempty, closed, and convex subset of a real Banach space E, and let $D : C \to C$ be a nonlinear map. The mapping $I - D$ is said to be demiclosed at zero if, for any sequence $\{x_n\} \subset C$, $x_n \rightharpoonup x$ and $(I-D)x_n \to 0 \implies x \in F(D)$.

Let $D : C \to int(domg)$ be a nonlinear mapping. Then

(i) a point $p \in C$ is called an asymptotic fixed point of D, if C contains a sequence $\{x_n\}$ which converges weakly to p and $\lim_{n\to\infty} \|Dx_n - x_n\| = 0$. We denote by $\hat{F}(D)$ the set of asymptotic fixed points of D.

(ii) D is called Bregman firmly nonexpansive if $\langle \Delta_g(Dx) - \Delta_g(Dy), Dx - Dy\rangle \leq \langle \Delta_g(x) - \Delta_g(y), Dx - Dy\rangle$ for all $x, y \in C$.

(iii) D is called Bregman strongly nonexpansive if $\Delta_g(p, Dx) \leq \Delta_g(p, x)$ for all $p \in \hat{F}(D)$ and $x \in C$.

(iv) D is called Bregman quasi-nonexpansive if $F(D) \neq \emptyset$ and $\Delta_g(p, Dx) \leq \Delta_g(p, x)$ for all $x \in C$ and $p \in F(D)$.

To solve EP (1.1), the following assumptions are also needed:

Assumption 2.1.

(A1) f is pseudomonotone, that is, for every $x, y \in C$, $f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0$ and $f(x, x) = 0$, for all $x \in C$.

(A2) The Bregman-Lipschitz type condition on C is satisfied by f, that is, there are two positive constants c_1 and c_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - c_1 \Delta_g(y, x) - c_2 \Delta_g(y, z), \ \forall\, x, y, z \in C,$$

where $g : E \to (-\infty, +\infty]$ is a Legendre function. With respect to g, c_1 and c_2 are referred to as Bregman-Lipschitz coefficients.

(A3) $f(x, .)$ is convex, lower semicontinuous, and subdifferentiable on C for all $x \in C$.

(A4) If $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ converge weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$ as $n \to \infty$, f is jointly weakly continuous on $C \times C$.

The Bregman projection from space E onto set C denoted by Π_C satisfies the property

$$\Delta_p(x, \Pi_C(x)) = \inf_{y \in C} \Delta_p(x, y), \ \forall\, x \in E.$$

Observe that $\Delta_p(\Pi_C x, y) + \Delta_p(x, \Pi_C x) \leq \Delta_p(x, y)$ for all y in set C; and $z = \Pi_C x$ if and only if

$$\langle y - z,\ J_E^p(x) - J_E^p(z)\rangle \leq 0, \ \forall\, y \in C; \tag{2.1}$$

Lemma 2.1. *[17, 23] Let E be a real p-uniformly convex and uniformly smooth Banach space, and $r > 0$ be a constant. Let $z, x_k \in E$ $(k = 1, 2, ..., N)$ and $\alpha_k \in (0, 1)$ with $\sum_{k=1}^N \alpha_k = 1$. Then,*

$$\Delta_p\Big(J_q^{E^*}\Big(\sum_{k=1}^N \alpha_k J_p^E(x_k)\Big), z\Big) \leq \sum_{k=1}^N \alpha_k \Delta_p(x_k, z) - \alpha_i \alpha_j g_r^*(\|J_p^E(x_i) - J_p^E(x_j)\|),$$

for all $i, j \in \{1, 2, ..., N\}$ and $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ being a strictly increasing function such that $g_r^*(0) = 0$.

For any $x^* \in C$, the subgradient of the convex function $g(x^*, .)$ at x^* is represented by $\partial_2 g(x^*, x^*)$, that is,

$$\begin{aligned}\partial_2 g(x^*, x^*) &= \{\phi \in E^* : g(x^*, v) \geq g(x^*, x^*) + \langle \phi, v - x^*\rangle, \ \forall\, v \in C\} \\ &= \{\phi \in E^* : g(x^*, v) \geq \langle \phi, v - x^*\rangle, \ \forall\, v \in C\}.\end{aligned}$$

Lemma 2.2. *[35] Let K be a nonempty and convex subset of a Banach space E. Let $g : K \to \mathbb{R}$ be a convex and subdifferentiable function on K. Then g attains its minimum at $x \in K$ if and only if $0 \in \partial g(x) + N_K(x)$, where $N_K(x)$ is the normal cone of K at x, that is,*

$$N_K(x) := \{\hat{x} \in E^* : \langle x - \varphi, \hat{x} \rangle \geq 0, \ \forall \ \varphi \in K\}.$$

Lemma 2.3. *[31] Let E be a real p-uniformly convex and uniformly smooth Banach space, $g : E \to \mathbb{R}$ be a strong coercive Bregman function and $V_p : E^* \times E \to [0, +\infty)$ be defined by*

$$V_p(x, x^*) = \frac{1}{p}\|x\|^p - \langle x, x^* \rangle + \frac{1}{q}\|x^*\|^q, \ \forall \ x \in E, x^* \in E^*.$$

Then, the following assertions hold:
(i) V_p is nonnegative and convex in the first variable.
(ii) $\Delta_p(x, J_q^{E^*}(x^*)) = V_p(x, x^*), \ \forall \ x \in E, \ x^* \in E^*$.
(iii) $V_p(x, x^*) + \langle J_q^{E^*}(x^*) - x, y^* \rangle \leq V_p(x, x^* + y^*), \forall \ x \in E, \ x^*, y^* \in E^*$.

Lemma 2.4. *[39] Let $q \geq 1$ and $r > 0$ be two fixed real numbers. Then, a Banach space E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g : \mathbb{R}^+ \to \mathbb{R}^*$, $g(0) = 0$ such that, for all $x, y \in B_r$ and $0 \leq \alpha < 1$,*

$$\|\alpha x + (1 - \alpha)y\|^q \leq \alpha\|x\|^q + (1 - \alpha)\|y\|^q - W_q(\alpha)g(\|x - y\|),$$

where $W_q(\alpha) := \alpha^q(1 - \alpha) + \alpha(1 - \alpha)^q$ and $B_r := \{x \in E : \|x\| \leq r\}$.

Lemma 2.5. *[41] Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E. Then the following two assertions are equivalent: (i) $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$; (ii) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.6. *[29] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{\Delta_f(x_n, x_0)\}$ is bounded, then $\{x_n\}$ is also bounded.*

Lemma 2.7. *[15] Let $a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n$ for all $n \geq 0$, where $\{a_n\}$ is a nonnegative real sequence, $\{\alpha_n\}$ is a sequence in $(0, 1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ being a real sequence. If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition: $\liminf_{k\to\infty}(a_{n_k+1} - a_{n_k}) \geq 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.8. *[7] Let C be a nonempty, closed, and convex subset of a uniformly convex and uniformly smooth Banach space E and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions $A_1 - A_4$ of Assumption 2.1. For the arbitrary sequences $\{u_n\} \subset C$ and $\{\lambda_n\} \subset (0, +\infty)$, let $\{z_n\}$ and $\{t_n\}$ be sequences generated by*

$$\begin{cases} z_n = argmin\{\lambda_n f_1(u_n, w) + \Delta_p(w, u_n) : \ w \in C\}, \\ t_n = argmin\{\lambda_n f_1(z_n, w) + \Delta_p(w, u_n) : \ w \in C\}. \end{cases}$$

Then, for all $x^* \in EP(f)$,

(1) $\langle J_p^{E_1} u_n - J_p^{E_1} z_n, w - z_n \rangle \leq \lambda_n[f_1(u_n, w) - f_2(u_n, z_n)]$;
(2) $\Delta_p(x^*, t_n) \leq \Delta_p(x^*, u_n) - (1 - \lambda_n c_1)\Delta_p(z_n, u_n) - (1 - \lambda_n c_2)\Delta_p(t_n, z_n)$.

3. The Algorithm

In this section, we present our proposed method and highlight some of its features. From now on, we assume that

Assumption 3.1.

(1) E is a p-uniformly convex real Banach space, which is also uniformly smooth.
(2) C is a nonempty, closed, and convex subset of E.
(3) $f : C \times C \to \mathbb{R}$ is a bifunction satisfying the $A_1 - A_4$ of Assumption 2.1.
(4) $D_j : E \to E,\ j = 1,2,\ldots,m$ is a finite family of Bregman quasi-nonexpansive mappings such that $I - D_j$ is demiclosed at zero for each $j = 1,2,\ldots,m$.
(5) The solution set $\Omega := \mathrm{EP}(f,C) \cap \bigcap_{j=1}^{m} \mathrm{Fix}(D_j) \neq \emptyset$.

In addition, let $\{\gamma_n\},\ \{\varepsilon_n\},\ \{\eta_{n,j}\}_{j=0}^{m}$ be positive sequences satisfying the following conditions:

(C1) $\{\gamma_n\} \in (0,1),\ \lim_{n\to\infty} \gamma_n = 0,\ \sum_{n=0}^{\infty} \gamma_n = \infty$.
(C2) $\{\varepsilon_n\} \in (0,1)$ such that $\lim\limits_{n\to\infty} \frac{\varepsilon_n}{\gamma_n} = 0$.
(C3) $\{\eta_{n,j}\} \in (0,1),\ \sum_{j=0}^{m} \eta_{n,j} = 1$ and $\liminf_{n\to\infty} \eta_{n,0}\eta_{n,j} > 0$.

We now present the algorithm. The sequence $\{x_n\}$ is generated as follows:

Algorithm 3.1. Calculation of the sequence $\{x_n\}$.

Initialization: Choose $x_0 \in C,\ u \in E,\ \rho_0 > 0,\ \mu \in (0,1),\ \theta \geq 3$ and set $n = 1$.

Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$, we choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n := \begin{cases} \min\left\{\frac{n-1}{n+\theta-1}, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \\ \frac{n-1}{n+\theta-1}, & \text{otherwise.} \end{cases} \tag{3.1}$$

Set

$$w_n = J^{-1}\Big(J(x_n) + \theta_n(J(x_{n-1}) - J(x_n))\Big). \tag{3.2}$$

Step 2: Compute

$$u_n = \arg\min_{z\in C}\{\rho_n f(w_n,z) + \Delta_p(z,w_n)\}.$$

If $w_n = u_n$, set $u_n = v_n$ and go to **Step 4**. Else, do **Step 3**.
Step 3: Compute

$$v_n = \arg\min_{z\in \zeta_n}\{\rho_n f(u_n,z) + \Delta_p(z,w_n)\},$$

where $\zeta_n = \{z \in E : \langle J(w_n) - \rho_n\varphi_n - J(u_n), z - u_n\rangle \leq 0\}$, and $\varphi_n \in \partial_2 f(w_n,u_n)$ satisfying $J(w_n) - \rho_n\varphi_n - J(u_n) \in N_C(u_n)$.

Step 4: Compute

$$z_n = J^{-1}\left(\eta_{n,0}J(v_n) + \sum_{j=1}^{m} \eta_{n,j}J(D_j v_n)\right).$$

Step 5: Calculate x_{n+1} and ρ_{n+1} as follows:

$$x_{n+1} = J^{-1}\left(\gamma_n J(u) + (1-\gamma_n)J(z_n)\right),$$

and

$$\rho_{n+1} = \begin{cases} \min\{\rho_n, \frac{\mu(\Delta_p(u_n,w_n)+\Delta_p(v_n,u_n))}{f(w_n,v_n)-f(w_n,u_n)-f(u_n,v_n)}\} \\ \text{if } f(w_n,v_n)-f(w_n,u_n)-f(u_n,v_n) > 0 \\ \rho_n, \text{ otherwise.} \end{cases} \tag{3.3}$$

Set $n = n+1$ and revert to **Step 1**.

We now highlight some of the features of our proposed method.

Remark 3.1. In Algorithm 3.1, $g : E \to \mathbb{R}$ is a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E such that $C \subset int\ domg$.

Remark 3.2. (i) The authors in ([7, 30]) introduced extragradient methods for solving pseudo-monotone EP, whose convergence depends on the Lipschitz coefficients c_1 and c_2 of the equilibrium bifunctions, which are usually unknown and extremely challenging to calculate or even estimate. In addition, each iteration of their approaches requires the solution of two strongly convex optimization problems over set C. We solve the second arg min issue over a constructible half-space ζ_n, but our approach is built in a way that its implementation is independent of the Lipschitz coefficients of the bifunction..

(ii) The authors in [13] investigated an extragradient algorithm for approximating the EP solution by using a line search strategy in the context of 2-uniformly convex Banach spaces. It is well known that the inner-loops used by the line search strategy may slow the pace of convergence of their scheme and lengthen computation times. The line-search method is less effective than our suggested Algorithm 3.1, which employs a straightforward self-adaptive step size methodology.

(iii) By using the Bregman distance method, we expand the findings of [12, 21, 40] on the subgradient extragradient procedure to p-uniformly convex Banach spaces.

Remark 3.3. From (3.1), it follows that

$$\lim_{n\to\infty} \frac{\theta_n}{\gamma_n}\|x_{n-1} - x_n\| = 0. \tag{3.4}$$

In fact, we obtain $\theta_n\|x_{n-1} - x_n\| \leq \varepsilon_n$, which follows from the above inequality and hypothesis (C2) that

$$\lim_{n\to\infty} \frac{\theta_n}{\gamma_n}\|x_{n-1} - x_n\| \leq \frac{\varepsilon_n}{\gamma_n} = 0.$$

Consequently, by the norm-to-norm continuity of J, it follows that

$$\lim_{n\to\infty} \frac{\theta_n}{\gamma_n}\|Jx_{n-1} - Jx_n\| = 0. \tag{3.5}$$

4. Convergence Analysis

We now present the strong convergence analysis of Algorithm 3.1. First, we establish the following necessary results.

Lemma 4.1. *The sequence $\{\rho_n\}$ generated by Algorithm 3.1 is well-defined and bounded.*

Proof. From (3.3), it is clear that $\rho_{n+1} \leq \rho_n$ for all $n \in \mathbb{N}$. This means that ρ_{n+1} is monotonically decreasing. Furthermore, we obtain from Assumption 2.1 that

$$c_1\Delta_g(u_n,w_n)+c_2\Delta_g(v_n,u_n) \geq f(w_n,v_n)-f(w_n,u_n)-f(u_n,v_n).$$

Hence

$$\frac{\mu\left(\Delta_g(u_n,w_n)+\Delta_g(v_n,u_n)\right)}{f(w_n,v_n)-f(w_n,u_n)-f(u_n,v_n)} \geq \frac{\mu\left(\Delta_g(u_n,w_n)+\Delta_g(v_n,u_n)\right)}{\max\{c_1,c_2\}(\Delta_g(u_n,w_n)+\Delta_g(v_n,u_n))} = \frac{\mu}{\max\{c_1,c_2\}}.$$

Therefore, we conclude that $\{\rho_n\}$ is bounded by $\min\{\rho_0, \frac{\mu}{\max\{c_1,c_2\}}\} > 0$, which implies that $\lim_{n\to\infty}\rho_n = \rho > 0$. □

Lemma 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 such that $\{x_n\}$ is bounded. Suppose conditions (C1) and (C2) hold. Then, for all $x^* \in \Omega$,*

$$\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\Big(\triangle_p(x^*,x_{n-1})-\triangle_p(x^*,x_n)\Big)=0.$$

Proof. Let $x^* \in \Omega$. Observe that

$$\begin{aligned}
&\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n)\\
&=\frac{1}{q}\|x_{n-1}\|^p-\langle x^*,J(x_{n-1})\rangle+\frac{1}{p}\|x^*\|^p-\Big(\frac{1}{q}\|x_n\|^p-\langle x^*,J(x_n)\rangle+\frac{1}{p}\|x^*\|^p\Big)\\
&=\frac{1}{q}\Big(\|x_{n-1}\|^p-\|x_n\|^p\Big)+\langle x^*,J(x_n)-J(x_{n-1})\rangle\\
&\leq\frac{1}{q}M\|x_{n-1}-x_n\|+\|J(x_n)-J(x_{n-1})\|\|x^*\|,
\end{aligned} \tag{4.1}$$

for some constant $M > 0$. Applying (3.4) and (3.5), we obtain from (4.1) that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\theta_n}{\gamma_n}\Big(\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n)\Big)\\
&\leq\lim_{n\to\infty}\Big(\frac{M}{q}\cdot\frac{\theta_n}{\gamma_n}\|x_{n-1}-x_n\|+\|x^*\|\frac{\theta_n}{\gamma_n}\|J(x_n)-J(x_{n-1})\|\Big)=0,
\end{aligned}$$

which is the required result. □

Lemma 4.3. *Let 3.1 hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 is bounded.*

Proof. Let $x^* \in \Omega$. We obtain from (3.2) that

$$\begin{aligned}
\Delta_p(x^*,w_n)&=\Delta_p\Big[x^*,J^{-1}\Big((1-\theta_n)J(x_n)+\theta_nJ(x_{n-1})\Big)\Big]\\
&\leq(1-\theta_n)\Delta_p(x^*,x_n)+\theta_n(x^*,x_{n-1}).
\end{aligned} \tag{4.2}$$

Since $v_n \in \zeta_n$, we obtain from Algorithm 3.1 that $\langle J(w_n) - \rho_n \varphi_n - J(u_n), v_n - u_n \rangle \leq 0$, which implies that

$$\langle J(w_n) - J(u_n), v_n - u_n \rangle \leq \rho_n \langle \varphi_n, v_n - u_n \rangle. \tag{4.3}$$

Also, since $\varphi_n \in \partial_2 f(w_n, u_n)$, we have that $f(w_n, z) - f(w_n, u_n) \geq \langle \varphi_n, z - u_n \rangle$ for all $z \in E$. In particular, we have

$$f(w_n, v_n) - f(w_n, u_n) \geq \langle \varphi_n, v_n - u_n \rangle. \tag{4.4}$$

Using (4.3) and (4.4), we obtain

$$\langle J(w_n) - J(u_n), v_n - u_n \rangle \leq \rho_n \{ f(w_n, v_n) - f(w_n, u_n) \}. \tag{4.5}$$

Furthermore, since $v_n = \arg\min_{z \in \zeta_n} \{ \rho_n f(u_n, z) + \Delta_g(z, w_n) \}$, we obtain from Lemma 2.2 that

$$0 \in \partial \left(\rho_n f(u_n, z) + \Delta_g(z, w_n) \right)(v_n) + N_{\zeta_n}(v_n),$$

which implies that there exist $\bar{\varphi}_n \in \partial_2 f(u_n, v_n)$ and $\phi \in N_{\zeta_n}(v_n)$, such that

$$\rho_n \bar{\varphi}_n + J(v_n) - J(w_n) + \phi = 0.$$

Furthermore, since $\phi \in N_{\zeta_n}(v_n)$, for all $z \in \zeta_n$, we have $\langle \phi, z - v_n \rangle \leq 0$. Therefore,

$$\rho_n \langle \bar{\varphi}_n, z - v_n \rangle \geq \langle J(w_n) - J(v_n), z - v_n \rangle, \ \forall \ z \in \zeta_n. \tag{4.6}$$

Moreover, $\bar{\varphi}_n \in \partial_2 f(u_n, v_n)$ implies that $f(u_n, z) - f(u_n, v_n) \geq \langle \bar{\varphi}_n, z - v_n \rangle$ for all $z \in \zeta_n$. Thus,

$$\rho_n \left(f(u_n, z) - f(u_n, v_n) \right) \geq \rho_n \langle \bar{\varphi}_n, z - v_n \rangle \ \forall \ z \in \zeta_n. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$\langle J(w_n) - J(v_n), z - v_n \rangle \leq \rho_n \left(f(u_n, z) - f(u_n, v_n) \right) \ \forall \ z \in \zeta_n. \tag{4.8}$$

Since $x^* \in \Omega$, then $x^* \in EP(f, C)$. Observe that $EP(f, C) \subset C \subset \zeta_n$. Hence, by letting $z = x^* \in \text{EP}(f, C)$ in (4.8), we have

$$\langle J(w_n) - J(v_n), x^* - v_n \rangle \leq \rho_n \left(f(u_n, x^*) - f(u_n, v_n) \right).$$

Since $f(x^*, u_n) \geq 0$, then by the pseudo-monotonicity of the bifunction f, we have $f(u_n, x^*) \leq 0$. Consequently,

$$\langle J(w_n) - J(v_n), x^* - v_n \rangle \leq -\rho_n f(u_n, v_n). \tag{4.9}$$

Adding (4.5) and (4.9) together gives

$$\begin{aligned} &\langle J(w_n) - J(v_n), x^* - v_n \rangle + \langle J(w_n) - J(u_n), v_n - u_n \rangle \\ &\leq \rho_n \{ f(w_n, v_n) - f(w_n, u_n) - f(u_n, v_n) \}. \end{aligned}$$

Applying Bregman three point identity, we have

$$\begin{aligned} \Delta_p(x^*, v_n) \leq \Delta_p(x^*, w_n) - \Delta_p(u_n, w_n) - \Delta_p(v_n, u_n) \\ + \rho_n \{ f(w_n, v_n) - f(w_n, u_n) - f(u_n, v_n) \}. \end{aligned}$$

Besides, we obtain from the definition of ρ_n that

$$\begin{aligned}
&\Delta_p(x^*, v_n)\\
&\leq \Delta_p(x^*, w_n) - \Delta_p(u_n, w_n) - \Delta_p(v_n, u_n) + \frac{\rho_n}{\rho_{n+1}}\rho_{n+1}\{f(w_n, v_n) - f(w_n, u_n) - f(u_n, v_n)\}\\
&\leq \Delta_p(x^*, w_n) - \left(1 - \frac{\rho_n}{\rho_{n+1}}\mu\right)\left(\Delta_p(u_n, w_n) + \Delta_p(v_n, u_n)\right). \qquad (4.10)
\end{aligned}$$

Since $\lim_{n\to\infty}\left(1 - \frac{\rho_n}{\rho_{n+1}}\mu\right) = 1 - \mu > 0$, then there exists $N \in \mathbb{N}$ such that $\left(1 - \frac{\rho_n}{\rho_{n+1}}\mu\right) > 0$ for all $n \geq N$. Hence, $\Delta_p(x^*, v_n) \leq \Delta_p(x^*, w_n)$ for all $n \geq N$. Also, from Algorithm 3.1 and the fact that D_j is quasi-nonexpansive for each $j \in \mathbb{N}$, we have

$$\begin{aligned}
\Delta_p(x^*, z_n) &= \Delta_p\left(x^*, J^{-1}\left(\eta_{n,0}J(v_n) + \sum_{j=1}^{m}\eta_{n,j}J(D_j v_n)\right)\right)\\
&\leq \eta_{n,0}\Delta_p(x^*, v_n) + \sum_{j=1}^{m}\eta_{n,j}\Delta_p(x^*, D_j v_n)\\
&\leq \Delta_p(x^*, v_n).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Delta_p(x^*, x_{n+1}) &\leq \gamma_n\Delta_p(x^*, u) + (1-\gamma_n)\Delta_p(x^*, z_n)\\
&\leq \gamma_n\Delta_p(x^*, u) + (1-\gamma_n)\Delta_p(x^*, w_n)\\
&\leq \gamma_n\Delta_p(x^*, u) + (1-\gamma_n)[(1-\theta_n)\Delta_p(x^*, x_n) + \theta_n(x^*, x_{n-1})]\\
&\leq \max\{\Delta_p(x^*, u)\}, \max\{\Delta_p(x^*, x_n), \Delta_p(x^*, x_{n-1})\}\\
&\vdots\\
&\leq \max\{\Delta_p(x^*, u)\}, \max\{\Delta_p(x^*, x_N), \Delta_p(x^*, x_{N-1})\}.
\end{aligned}$$

Therefore, $\{\Delta_p(x^*, x_n)\}$ is bounded. From Lemma 2.6, one sees that $\{x_n\}$ is also bounded. Consequently, $\{z_n\}$, $\{v_n\}$ $\{u_n\}$ and $\{w_n\}$ are all bounded. □

Lemma 4.4. *Let $r = \sup_{n\in\mathbb{N}}\{\|J(v_n)\|, \|J(D_j v_n)\|\}$ and let $W_q : E^* \to \mathbb{R}$ be the gauge of uniform convexity of the conjugate function g^*. It follows that*

$$\Delta_p(x^*, z_n) \leq \Delta_p(x^*, v_n) - \frac{W_q(\eta_{n,j})}{q}g\left(\|J(v_n) - J(D_j v_n)\|\right), \qquad (4.11)$$

where $W_q(\eta_{n,j}) = (\eta_{n,0})^q\sum_{j=1}^{m}\eta_{n,j} + \eta_{n,0}(\sum_{j=1}^{m}\eta_{n,j})^q$.

Proof. Indeed, we obtain from Algorithm 3.1 that

$$\Delta_p(x^*, z_n) = V_p\left(x^*, \eta_{n,0}J(v_n) + \sum_{j=1}^{m}\eta_{n,j}J(D_j v_n)\right).$$

Furthermore, we see from Lemma 2.3 and Lemma 2.4 that

$$\begin{aligned}
\Delta_p(x^*, z_n) &\leq \frac{1}{p}\|x^*\|^p - \eta_{n,0}\langle x^*, J(v_n)\rangle - \sum_{j=1}^{m}\eta_{n,j}\langle x^*, J(D_j v_n)\rangle \\
&\quad + \frac{1}{q}\eta_{n,0}\|J(v_n)\|^q + \frac{1}{q}\sum_{j=1}^{m}\eta_{n,j}\|J(D_j v_n)\|^q - \frac{W_q(\eta_{n,j})}{q}g\big(\|J(v_n) - J(D_j v_n)\|\big) \\
&= \frac{1}{p}\eta_{n,0}\|x^*\|^p + \frac{1}{p}\sum_{j=1}^{m}\eta_{n,j}\|x^*\|^p - \eta_{n,0}\langle x^*, J(v_n)\rangle - \sum_{j=1}^{m}\eta_{n,j}\langle x^*, J(D_j v_n)\rangle \\
&\quad + \frac{1}{q}\eta_{n,0}\|J(v_n)\|^q + \frac{1}{q}\sum_{j=1}^{m}\eta_{n,j}\|J(D_j v_n)\|^q - \frac{W_q(\eta_{n,j})}{q}g\big(\|J(v_n) - J(D_j v_n)\|\big) \\
&= \eta_{n,0}\left\{\frac{1}{p}\|x^*\|^p - \langle x^*, J(v_n)\rangle + \frac{1}{q}\|J(v_n)\|^q\right\} \\
&\quad + \sum_{j=1}^{m}\eta_{n,j}\left\{\frac{1}{p}\|x^*\|^p - \langle x^*, J(D_j v_n)\rangle + \frac{1}{q}\|J(D_j v_n)\|^q\right\} \\
&\quad - \frac{W_q(\eta_{n,j})}{q}g\big(\|J(v_n) - J(D_j v_n)\|\big) \\
&= \eta_{n,0}\Delta_p(x^*, v_n) + \sum_{j=1}^{m}\eta_{n,j}\Delta_p(x^*, D_j v_n) - \frac{W_q(\eta_{n,j})}{q}g\big(\|J(v_n) - J(D_j v_n)\|\big).
\end{aligned}$$

Since D_j is Bregman quasi-nonexpansive for each $j \in \mathbb{N}$, we obtain

$$\begin{aligned}
\Delta_p(x^*, z_n) &\leq \eta_{n,0}\Delta_p(x^*, v_n) + \sum_{j=1}^{m}\eta_{n,j}\Delta_p(x^*, v_n) - \frac{W_q(\eta_{n,j})}{q}g\big(\|J(v_n) - J(D_j v_n)\|\big) \\
&= \Delta_p(x^*, v_n) - \frac{W_q(\eta_{n,j})}{q}g\big(\|J(v_n) - J(D_j v_n)\|\big).
\end{aligned}$$

□

Next, we prove the strong convergence theorem for the sequence $\{x_n\}$ generated by our proposed algorithm.

Theorem 4.1. *The iterative sequence $\{x_n\}$ generated by Algorithm 3.1 under Assumption 3.1 strongly converges to $x^* \in \Omega$, where $x^* = Proj_{\Omega}^{f}(u)$.*

Proof. Suppose $x^* = \mathrm{Proj}_{\Omega}^{f}(u)$. Then from Algorithm 3.1 and by applying Lemma 2.3 (iii), we have

$$\begin{aligned}
&\Delta_p(x^*, x_{n+1}) \\
&= V_p\left(x^*, \gamma_n J(u) + (1-\gamma_n)J(z_n)\right) \\
&\leq V_p\Big(x^*, \gamma_n J(u) + (1-\gamma_n)J(z_n) - \gamma_n\big(J(u) - J(x^*)\big)\Big) + \gamma_n\langle J(u) - J(x^*), x_{n+1} - x^*\rangle \\
&= V_p\Big(x^*, \gamma_n J(x^*) + (1-\gamma_n)J(z_n)\Big) + \gamma_n\langle J(u) - J(x^*), x_{n+1} - x^*\rangle \\
&\leq (1-\gamma_n)\Delta_p(x^*, z_n) + \gamma_n\langle J(u) - J(x^*), x_{n+1} - x^*\rangle.
\end{aligned} \tag{4.12}$$

Applying (4.11) in (4.12), we obtain

$$\begin{aligned}\Delta_p(x^*,x_{n+1}) \leq (1-\gamma_n)\left[\Delta_p(x^*,v_n) - \frac{W_q(\eta_{n,j})}{q} g\big(\|J(v_n)-J(D_j v_n)\|\big)\right] \\ + \gamma_n \langle J(u)-J(x^*), x_{n+1}-x^*\rangle.\end{aligned} \tag{4.13}$$

Furthermore, by substituting (4.2) and (4.10) into (4.13), we have

$$\begin{aligned}&\Delta_p(x^*,x_{n+1}) \leq (1-\gamma_n) \\ &\left[\Delta_p(x^*,w_n) - \left(1-\frac{\rho_n}{\rho_{n+1}}\mu\right)\Big(\Delta_p(u_n,w_n)+\Delta_p(v_n,u_n)\Big) - \frac{W_q(\eta_{n,j})}{q} g(\|J(v_n)-J(D_j v_n)\|)\right] \\ &+ \gamma_n \langle J(u)-J(x^*), x_{n+1}-x^*\rangle \\ &\leq (1-\gamma_n)\Delta_p(x^*,x_n) - (1-\gamma_n)\left(1-\frac{\rho_n}{\rho_{n+1}}\mu\right)\Big(\Delta_p(u_n,w_n)\Delta_p(v_n,u_n)\Big) \\ &- (1-\gamma_n)\frac{W_q(\eta_{n,j})}{q} g(\|J(v_n)-J(D_j v_n)\|) + \gamma_n \Psi_n,\end{aligned} \tag{4.14}$$

where $\Psi_n := \left(\frac{\theta_n}{\gamma_n}\Big(\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n)\Big) + \langle J(u)-J(x^*), x_{n+1}-x^*\rangle\right)$. Thus

$$(1-\gamma_n)\left(1-\frac{\rho_n}{\rho_{n+1}}\mu\right)\Big(\Delta_p(u_n,w_n)+\Delta_p(v_n,u_n)\Big) \leq \Delta_p(x^*,x_n) - \Delta_p(x^*,x_{n+1}) + \gamma_n\Psi_n, \tag{4.15}$$

It also follows from (4.14) that

$$(1-\gamma_n)\frac{W_q(\eta_{n,j})}{q} g(\|J(v_n)-J(D_j v_n)\|) \leq \Delta_p(x^*,x_n) - \Delta_p(x^*,x_{n+1}) + \gamma_n\Psi_n. \tag{4.16}$$

Moreover, we see that

$$\Delta_p(x^*,x_{n+1}) \leq (1-\gamma_n)\Delta_p(x^*,x_n) + \gamma_n\Psi_n. \tag{4.17}$$

We now show that the sequence $\{x_n\}$ converges to x^*. Let $a_n := \Delta_p(x_n,x^*)$. It is easy to see that (4.17) satisfies the inequality $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\Psi_n$. To apply Lemma 2.7, it is enough to show that $\limsup_{k\to\infty}\Psi_{n_k} \leq 0$ for every subsequence $\{\Delta_p(x^*,x_{n_k})\}$ of $\{\Delta_p(x^*,x_n)\}$ satisfying the relation

$$\liminf_{k\to\infty}\big(\Delta_p(x^*,x_{n_k+1}) - \Delta_p(x^*,x_{n_k})\big) \geq 0. \tag{4.18}$$

Now, we assume that $\{\Delta_p(x^*,x_{n_k})\}$ is a subsequence of $\{\Delta_p(x^*,x_n)\}$ such that (4.18) holds. Then, from (4.15), (4.18), and condition (C1), we have that

$$\begin{aligned}&\limsup_{k\to\infty}(1-\gamma_{n_k})\left(1-\frac{\rho_{n_k}}{\rho_{n_k+1}}\mu\right)\big(\Delta_p(u_{n_k},w_{n_k})+\Delta_p(v_{n_k},u_{n_k})\big) \\ &\leq \limsup_{k\to\infty}\Big(\Delta_p(x^*,x_{n_k}) - \Delta_p(x^*,x_{n_k+1}) + \gamma_{n_k}\Psi_{n_k}\Big) \\ &= \limsup_{k\to\infty}\Big(\Delta_p(x^*,x_{n_k}) - \Delta_p(x^*,x_{n_k+1})\Big) \\ &\leq -\liminf_{k\to\infty}\Big(\Delta_p(x^*,x_{n_k+1}) - \Delta_p(x^*,x_{n_k})\Big) \leq 0.\end{aligned}$$

It follows that

$$\limsup_{k\to\infty}\left(1-\frac{\rho_{n_k}}{\rho_{n_k+1}}\mu\right)\left(\Delta_p(u_{n_k},w_{n_k})+\Delta_p(v_{n_k},u_{n_k})\right)=0. \tag{4.19}$$

Since $\lim_{k\to\infty}\rho_{n_k}$ exists, then $\lim_{n\to\infty}\frac{\rho_{n_k}}{\rho_{n_k+1}}=1$. Using the fact that $\mu\in(0,1)$, it follows from (4.19) that

$$\lim_{k\to\infty}\Delta_p(u_{n_k},w_{n_k})=\lim_{k\to\infty}\Delta_p(v_{n_k},u_{n_k})=0.$$

By Lemma 2.5, we obtain

$$\lim_{k\to\infty}\|u_{n_k}-w_{n_k}\|=0=\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\|. \tag{4.20}$$

Consequently, we have

$$\lim_{k\to\infty}\|v_{n_k}-w_{n_k}\|\leq\lim_{k\to\infty}\|v_{n_k}-u_{n_k}\|+\lim_{k\to\infty}\|u_{n_k}-w_{n_k}\|=0. \tag{4.21}$$

Similarly, from (4.16) and (4.18), we have

$$\begin{aligned}
&\limsup_{k\to\infty}(1-\gamma_{n_k})\frac{W_q(\eta_{n_k,j})}{q}g(\|J(v_{n_k})-J(D_jv_{n_k})\|)\\
&\leq\limsup_{k\to\infty}\left(\Delta_p(x^*,x_{n_k})-\Delta_p(x^*,x_{n_k+1})+\gamma_{n_k}\Psi_{n_k}\right)\\
&=\limsup_{k\to\infty}\left(\Delta_p(x^*,x_{n_k})-\Delta_p(x^*,x_{n_k+1})\right)\\
&\leq-\liminf_{k\to\infty}\left(\Delta_p(x^*,x_{n_k+1})-\Delta_p(x^*,x_{n_k})\right)\leq 0.
\end{aligned}$$

Therefore,

$$\limsup_{k\to\infty}(1-\gamma_{n_k})\frac{W_q(\eta_{n_k,j})}{q}g(\|J(v_{n_k})-J(D_jv_{n_k})\|)=0.$$

By condition (C3), we have $\lim_{k\to\infty}g\|J(v_{n_k})-J(D_jv_{n_k})\|=0$, $j=1,2,\ldots,m$. By the property of g and the fact that J^{-1} is norm-to-norm continuous on bounded subsets of E^*, we have

$$\lim_{k\to\infty}\|v_{n_k}-D_jv_{n_k}\|=0,\quad j=1,2,\ldots,m. \tag{4.22}$$

Moreover, by applying condition (C3) and (4.22) we obtain

$$\lim_{k\to\infty}\|Jz_{n_k}-Jv_{n_k}\|\leq\lim_{k\to\infty}\left(\eta_{n_k,0}\|J(v_{n_k})-J(v_{n_k})\|+\sum_{j=1}^{m}\eta_{n_k,j}\|J(D_jv_{n_k})-J(v_{n_k})\|\right)=0.$$

By the norm-to-norm continuity of J^{-1} on bounded subsets of E^*, we arrive at

$$\lim_{k\to\infty}\|z_{n_k}-v_{n_k}\|=0. \tag{4.23}$$

From (4.21) and (4.23), we have

$$\lim_{k\to\infty}\|z_{n_k}-w_{n_k}\|=0. \tag{4.24}$$

Recall that $w_{n_k} = J^{-1}\Big[J(x_{n_k}) + \theta_{n_k}\big(J(x_{n_k-1}) - J(x_{n_k})\big)\Big]$. By using (3.5), we obtain

$$\lim_{k\to\infty} \|J(w_{n_k}) - J(x_{n_k})\| = \lim_{k\to\infty} \gamma_{n_k} \cdot \frac{\theta_{n_k}}{\gamma_{n_k}} \|J(x_{n_k-1}) - J(x_{n_k})\| = 0. \tag{4.25}$$

From the norm-to-norm continuity of J^{-1} on bounded subsets of E^*, we have

$$\lim_{k\to\infty} \|w_{n_k} - x_{n_k}\| = 0. \tag{4.26}$$

From (4.20), (4.23), (4.24) and (4.26) we obtain

$$\lim_{k\to\infty} \|z_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|v_{n_k} - x_{n_k}\| = \lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0. \tag{4.27}$$

Thus $\lim_{k\to\infty} \|J(x_{n_k+1}) - J(z_{n_k})\| = 0$. In view of the norm-to-norm continuity of J^{-1} on bounded subsets of E^*, we have $\lim_{k\to\infty} \|x_{n_k+1} - z_{n_k}\| = 0$, which together with (4.27) yields

$$\lim_{k\to\infty} \|x_{n_k+1} - x_{n_k}\| = 0. \tag{4.28}$$

Since $\{x_n\}$ is bounded, then $w_\omega(x_n)$ is nonempty. Let $\hat{x} \in w_\omega(x_n)$ be an arbitrary element. Then, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to $\hat{x} \in E$. It follows from (4.27) that $\{u_{n_k}\}$ converges weakly to $\hat{x} \in E$. In view of $u_{n_k} = \arg\min_{z\in C}\{\rho_{n_k} f(w_{n_k}, z) + \Delta_p(z, w_{n_k})\}$, it follows from Lemma 2.2 that

$$0 \in \partial\Big(\rho_{n_k} f(w_{n_k}, z) + \Delta_p(z, w_{n_k})\Big)(u_{n_k}) + N_C(u_{n_k}),\ z \in C,$$

which means that

$$\rho_{n_k}\varphi_{n_k} + J(u_{n_k}) - J(w_{n_k}) + \phi = 0,\ \text{where } \phi \in N_C(u_{n_k}). \tag{4.29}$$

Observe that $\langle \phi, z - u_{n_k}\rangle \le 0$ for all $z \in C$. Therefore, it follows from (4.29) that

$$\langle \rho_{n_k}\varphi_{n_k}, z - u_{n_k}\rangle + \langle \phi, z - u_{n_k}\rangle = \langle J(w_{n_k}) - J(u_{n_k}), z - u_{n_k}\rangle,$$

Consequently, we have

$$\rho_{n_k}\langle \varphi_{n_k}, z - u_{n_k}\rangle \ge \langle J(w_{n_k}) - J(u_{n_k}), z - u_{n_k}\rangle. \tag{4.30}$$

Furthermore, since $\varphi_n \in \partial_2 f(w_{n_k}, u_{n_k})$, we have

$$f(w_{n_k}, z) - f(w_{n_k}, u_{n_k}) \ge \langle \varphi_{n_k}, z - u_{n_k}\rangle,\ \forall\, z \in C. \tag{4.31}$$

Combining (4.30) and (4.31), we have

$$\rho_{n_k}\Big(f(w_{n_k}, z) - f(w_{n_k}, u_{n_k})\Big) \ge \langle J(w_{n_k}) - J(u_{n_k}), z - u_{n_k}\rangle,\ \forall\, z \in C. \tag{4.32}$$

Since $\lim_{k\to\infty} \|w_{n_k} - u_{n_k}\| = 0$ and J is uniformly continuous on bounded subsets of E, we have $\lim_{k\to\infty} \|J(w_{n_k}) - J(u_{n_k})\| = 0$. Consequently, passing limit $k \to \infty$ in (4.32) and applying condition (A4), (4.26), we obtain $f(\hat{x}, z) \ge 0$ for all $z \in C$, which implies that $\hat{x} \in \mathrm{EP}(f, C)$. Also, since $\lim_{k\to\infty} \|v_{n_k} - D_j v_{n_k}\| = 0$, $j = 1, 2, \ldots, m$ and $v_{n_k} \rightharpoonup \hat{x}$ by (4.27), we see by the demiclosedness of $I - D_j$ that $\hat{x} \in F(D_j)$ for all $j = 1, 2, \ldots, m$. Therefore, it follows that $\hat{x} \in \cap_{j=1}^m F(D_j)$. Since $\hat{x} \in w_\omega(x_n)$ is arbitrary, we have $w_\omega(x_n) \subset \mathrm{EP}(f, C) \cap \bigcap_{j=1}^m F(D_j)$. To complete the proof, we need to show that $\lim_{k\to\infty} \Psi_{n_k} \le 0$. First, we show that $\limsup_{k\to\infty} \langle J(u) - J(x^*), x_{n_{k+1}} - x^*\rangle \le 0$.

By the boundedness of $\{x_{n_k}\}$, we see that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup q$ and

$$\lim_{j\to\infty}\langle J(u)-J(x^*),x_{n_{k_j}}-x^*\rangle = \limsup_{k\to\infty}\langle J(u)-J(x^*),x_{n_k}-x^*\rangle.$$

Since $x^* = \text{Proj}^f_\Omega(u)$, then, by (2.1) and (4.28), we obtain

$$\begin{aligned}
&\limsup_{k\to\infty}\langle J(u)-J(x^*),x_{n_{k+1}}-x^*\rangle \\
&\quad= \limsup_{k\to\infty}\langle J(u)-J(x^*),x_{n_k}-x^*\rangle + \limsup_{k\to\infty}\langle J(u)-J(x^*),x_{n_{k+1}-x_{n_k}}\rangle \\
&\quad= \lim_{j\to\infty}\langle J(u)-J(x^*),x_{n_{k_j}}-x^*\rangle \\
&\quad= \langle J(u)-J(x^*),q-x^*\rangle \le 0. \qquad (4.33)
\end{aligned}$$

Recall that

$$\Psi_n = \left(\frac{\theta_n}{\gamma_n}(1-\gamma_n)(\Delta_p(x^*,x_{n-1})-\Delta_p(x^*,x_n)) + \langle J(u)-J(x^*),x_{n+1}-x^*\rangle\right).$$

By Lemma 4.2 and (4.33), we have that $\limsup_{k\to\infty}\Psi_{n_k}\le 0$. Next, Lemma 2.7 concludes that $\Delta_p(x^*,x_n)\to 0$ as $n\to\infty$. By Lemma 2.5, it follows that $\lim\|x_n-x^*\|=0$. Hence, $\{x_n\}$ converges strongly to $\hat{x}\in\Omega$ as required. □

Next, we apply our result to study a variational inequality problem (VIP). Let $A: C\to E^*$ be an operator. The VIP is formulated as finding a point

$$q^*\in C \text{ such that } \langle \bar{x}-q^*,A(q^*)\rangle\ge 0,\ \forall\,\bar{x}\in C. \qquad (4.34)$$

The solution set of VIP (4.34) is denoted by VIP(C,A).

Recall that the following definitions for mapping $A: C\to E^*$.

(D1) A is said to be pseudo-monotone; that is, for all x,y in C,

$$\langle Ax,y-x\rangle\ge 0 \Longrightarrow \langle Ay,y-x\rangle\ge 0,$$

(D2) A is said to be K-Lipschitz continuous with respect to Δ_p, that is, there exists a constant $K>0$ such that $\Delta_p(Ax,Ay)\le K\Delta_p(x,y)$ for all $x,y\in C$,

(D3) A is said to be sequentially weakly continuous, that is, for any sequence $\{x_n\}\subset C$, $x_n\rightharpoonup x\in C$ implies that $Ax_n\rightharpoonup Ax\in E^*$.

Lemma 4.5. *[7] Let C be a nonempty, closed, and convex subset of a reflexive Banach space E, and let $A: C\to E^*$ be a nonlinear mapping. Then,*

$$\Pi_C\left(J_q^{E^*}[J_p^E(x)-\theta A(y)]\right) = \arg\min_{z\in C}\{\theta\langle z-y,A(y)\rangle+\Delta_p(z,x)\},$$

for all $x\in E$, $y\in C$ and $\theta\in(0,+\infty)$.

If we take $f(x,y)=\langle A(x),y-x\rangle$ for all $x,y\in C$, then we find by $\varphi_n\in\partial_2 f(w_n,u_n)$ that

$$f(w_n,z)-f(w_n,u_n)\ge\langle\varphi_n,z-u_n\rangle,\ \forall\,z\in E,$$

which implies that $\langle A(w_n),z-w_n\rangle-\langle A(w_n),u_n-w_n\rangle\ge\langle\varphi_n,z-u_n\rangle$. Simplifying the last inequality, we obtain $\langle A(w_n)-\varphi_n,z-u_n\rangle\ge 0$ for all $z\in E$. From this, it follows that

$$\langle J(w_n)-\rho_n A(w_n)-J(u_n),z-u_n\rangle\le\langle J(w_n)-\rho_n\varphi_n-J(u_n),z-u_n\rangle\le 0,\ \forall\,z\in\zeta_n\subset E.$$

Now, taking $f(x,y) = \langle A(x), y-x\rangle$ for all $x,y \in C$, one sees that bifunction f satisfies the conditions (A1)-(A4) of Assumption 2.1 (see [7]). Hence, by applying Theorem 4.1 and Lemma 4.5, we obtain the following result for approximating a common solution of pseudo-monotone variational inequality problems and the common fixed point problem of a finite family of Bregman quasi-nonexpansive mappings in p-uniformly convex real Banach space, which is also uniformly smooth.

Theorem 4.2. *Let E be a p-uniformly convex real Banach space, which is also uniformly smooth, and let C be a nonempty, closed, and convex subset of E. Let $A : C \to E^*$ be a mapping satisfying conditions (D1)-(D3), and let $D_j : E \to E,\ j = 1,2,\ldots,m$ be a finite family of Bregman quasi-nonexpansive mappings such that $I - D_j$ is demiclosed at zero for each j. Suppose that conditions (C1)-(C3) hold and $\Gamma := VI(C,A) \cap \bigcap_{j=1}^{m} F(D_j) \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by Algorithm (4.1) below converges strongly to $x^* \in \Gamma$, where $x^* = Proj_{\Gamma}^{f}(u)$.*

Algorithm 4.1.

Choose $x_0 \in C,\ u \in E,\ \rho_0 > 0,\ \mu \in (0,1)$ and set $n = 1$. Given the iterates x_{n-1} and x_n, we have

$$\begin{cases} w_n = J^{-1}\Big(J(x_n) + \theta_n(J(x_{n-1}) - J(x_n))\Big) \\ u_n = \Pi_C[J^{-1}(J(w_n) - \rho_n A(w_n))] \\ \zeta_n = \{z \in E : \langle J(w_n) - \rho_n A(w_n) - J(u_n), z - u_n\rangle \leq 0\} \\ v_n = \Pi_{\zeta_n}[J^{-1}(J(w_n) - \rho_n A(u_n))] \\ z_n = J^{-1}\Big(\eta_{n,0}J(v_n) + \sum_{j=1}^{m} \eta_{n,j}J(D_j v_n)\Big) \\ x_{n+1} = J^{-1}\Big(\gamma_n J(u) + (1-\gamma_n)J(z_n)\Big), \end{cases} \tag{4.35}$$

where the sequence of step sizes $\{\rho_n\}$ is defined by

$$\rho_{n+1} = \begin{cases} \min\{\rho_n, \frac{\mu(\Delta_p(u_n,w_n)+\Delta_p(v_n,u_n))}{\langle Aw_n - Au_n, v_n - u_n\rangle}\} \\ \text{if } \langle Aw_n - Au_n, v_n - u_n\rangle > 0 \\ \rho_n, \text{ otherwise,} \end{cases}$$

and θ_n is defined by (3.1).

5. Numerical Experiments

In this section, we provide some numerical experiments to show the performance of our proposed iterative method, Algorithm 3.1 and compare it with some related methods in the literature.

In the numerical experiments, for our Algorithm 3.1, we consider the case for which $m = 5$ and choose $\mu = 0.9,\ \gamma_n = \frac{1}{(2n+1)},\ \eta_{n,0} = \frac{n}{2n+1}, \eta_{n,j} = \frac{n+1}{5(2n+1)},\ j = 1,2,\ldots,5$. In Appendix 7.1, we choose $\lambda_n = 0.3,\ \beta_n = \frac{n}{2n+1}$. In Appendix 7.2, we choose $\xi = 0.1,\ v = 0.4$ and in Appendix 7.3, we take $N = 1$. In Tables 1, 2 and 2, Iter. means the number of iterations while CPU means

the CPU time in seconds. Firstly, we consider the following two examples in finite dimensional spaces.

Example 5.1. Let the bifunction f be defined by $f(x,y) = [F(x)]^T (y-x)$, where $F(x) = Mx + P(x)$ with M an $p \times p$ symmetric semidefinite matrix and P is defined by:

$$P(x) = \arg\min \left[\frac{\|y\|^4}{4} + \frac{1}{2}\|y-x\|^2 : y \in \mathbb{R}^p \right].$$

Let $C = \{x \in \mathbb{R}^p : Ax \leq b\}$, where $A \in \mathbb{R}^{q\times p}$ and $b \in \mathbb{R}^q$ with $q = 10$. The bifunction f is pseudo-monotone and satisfies the conditions (A1)-(A4) of Assumption A (see [33]). In addition, for $j = 1,2,...m$, let $D_j : \mathbb{R}^p \to \mathbb{R}^p$ be defined by $D_j(x) = \frac{x}{3j} = T_j(x)$. It is easy to verify that D_j is Bregman quasi-nonexpansive for each $j = 1,2,...,m$. We choose $u = x_1 = (1,1,...,1)$, $x_0 = 2u \in C$ and different cases of p.

Example 5.2. Consider the Nash-Cournot oligopolistic equilibrium model [25]. Let the bifunction $f \in \mathbb{R}^m$ be defined as $f(x,y) = \langle Px + Qy + r, y - x\rangle$, where r is a vector in $\mathbb{R}^p$ and P, Q are two matrices of order p such that Q is symmetric positive semidefinite and $Q - P$ is symmetric negative semidefinite. We define the set C such that $C = \{x \in \mathbb{R}^p : Ax \leq b\}$, where $A \in \mathbb{R}^{q\times p}$ is a random matrix and $b \in \mathbb{R}^q$ with $q = 10$. It is easy to see that f is pseudo-monotone and satisfies conditions (A1)-(A4) of Assumption 2.1 with the Lipschitz constants $c_1 = c_2 = \frac{\|P-Q\|}{2}$. For the optimization program in Algorithm 3.1, we have the following steps:

$$u_n = \arg\min_{z\in C}\{\frac{1}{2}z^T H_n z + b_n^T z\},$$

where $H_n = 2\rho_n Q + I$ and $b_n = \rho_n[(P-Q)w_n + r] - w_n$, and

$$v_n = \arg\min_{z\in \zeta_n}\left[\frac{1}{2}z^T \bar{H}_n z + \bar{b}_n^T z\right], \tag{5.1}$$

where $\bar{H}_n = 2\rho_n Q + I$ and $\bar{b}_n = \rho_n[(P-Q)w_n + r] - w_n$. Also, since $\varphi_n \in \partial_2 f(w_n, u_n)$, we see that $\varphi_n = 2Qu_n + (P-Q)w_n + r$. Furthermore, letting

$$a_n = (I - \rho_n(P-Q))w_n - (2\rho_n Q + I)u_n - \rho_n r,$$

we see that $\zeta_n = \{x \in \mathbb{R}^p : \langle a_n, x\rangle \leq \langle a_n, u_n\rangle\}$. Equation (5.1) is a quadratic convex program which can be solved efficiently. In addition, for $j = 1,2,...m$, let $D_j : \mathbb{R}^p \to \mathbb{R}^p$ by $D_j(x) = \frac{x}{5j} = T_j$. It is easy to verify that D_j is Bregman quasi-nonexpansive for each $j = 1,2,...,m$. Also, r, P and Q are selected randomly and $u = x_0 = x_1 = (1,1,...,1) \in C$ with different cases of p.

We next consider the following example in infinite dimensional spaces.

Example 5.3. Let $E = \ell_2(\mathbb{R})$ be the linear spaces whose elements are all 2-summable sequences $\{x_i\}_{i=1}^\infty$, $\ell_2 = \{x : x = (x_1, x_2, ..., x_i, ...)$, $x_i \in \mathbb{R}$ and $\sum_{i=1}^\infty |x_i|^2 < \infty\}$, with inner product $\langle .,.\rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ and norm $\|.\| : \ell_2 \to \mathbb{R}$ defined by

$$\langle x, y\rangle = \sum_{i=1}^\infty x_i y_i \text{ and } \|x\| = \left(\sum_{i=1}^\infty |x_i|^2\right)^2, \text{ where } x = \{x_i\}_{i=1}^\infty,\ y = \{y_i\}_{i=1}^\infty \in \ell_2.$$

Let $C = \{x \in E : \|x\| \leq 1\}$. Define the bifuntion $f : C \times C \to \mathbb{R}$ by $f(x,y) = (3 - \|x\|)\langle x, y - x\rangle$ for all $x, y \in C$. It can easily be verified that f is pseudo-monotone and satisfies the conditions (A1)-(A4) of Assumption A with Lipschitz constant $c_1 = c_2 = \frac{5}{2}$. Also, for $j = 1,2,...m$, we define $D_j : \ell_2 \to \ell_2$ by $D_j(x) = \frac{x}{2j} = T_j(x)$. It is easy to verify that D_j is Bregman quasi-nonexpansive for each $j = 1,2,...,m$.

We test Example 5.1, Example 5.2 ,and Example 5.3 under the following experiments.

Experiment 5.1. In this experiment, we check the behavior of our method by fixing the other parameters and varying θ in Example 5.1. We do this to check the effects of this parameter and the sensitivity of our method on it. We choose $q_n = \frac{n}{2n+1}u$, $g_n = \frac{n}{2n+1}u$ and $p \in \{5,10,15,20\}$. Using $\|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion, we plot the graphs of $\|x_{n+1} - x_n\|$ against the number of iterations in each case. The numerical results are reported in Fig. 1 and Table 1.

Experiment 5.2. In this experiment, we check the behavior of our method by fixing the other parameters and varying ε_n in Example 5.2. We do this to check the effects of this parameter and the sensitivity of our method on it. Consider

$$\varepsilon_n \in \{\frac{1}{(2n+1)^3}, \frac{2}{(2n+5)^3}, \frac{1}{(2n+1)^4}, \frac{2}{(2n+5)^4}, \frac{1}{(2n+1)^5}\},$$

which satisfies Assumption 3.1 (5)-(C2). Choose $q_n = \frac{n}{2n+1}u$, $g_n = \frac{n}{2n+1}u$, and $p \in \{5,10,20,30\}$. Using $\|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion, we plot the graphs of $\|x_{n+1} - x_n\|$ against the number of iterations in each case. The numerical results are reported in Fig. 2 and Table 2.

Experiment 5.3. In this experiment, we check the behavior of our method by fixing the other parameters and varying ρ in Example 5.3. We do this to check the effects of this parameter and the sensitivity of our method on it. We consider $\rho \in \{0.5, 1.0, 1.5, 2.0, 2.5\}$. Using $\|x_{n+1} - x_n\| < 10^{-4}$ as the stopping criterion, we plot the graphs of $\|x_{n+1} - x_n\|$ against the number of iterations in each case. We choose $u = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$, $q_n = \frac{n}{2n+1}u$, $g_n = \frac{n}{2n+1}u$ and consider different cases of initial values x_0 and x_1 as follows:

Case 1 : $x_0 = (\frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \cdots)$, $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.
Case 2 : $x_0 = (2, 1, \frac{1}{2}, \cdots)$, $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.
Case 3 : $x_0 = (\frac{1}{4}, \frac{1}{16}, \frac{1}{64}, \cdots)$, $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.
Case 4 : $x_0 = (\frac{7}{9}, \frac{7}{18}, \frac{7}{36}, \cdots)$, $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.

The numerical results are reported in Fig. 3 and Table 3.

Table 1. Numerical results for Example 5.1 (Experiment 5.1).

Cases		App. 7.1	App. 7.2	App. 7.3	Alg. 3.1 ($\theta = 3$)	Alg. 3.1 ($\theta = 6$)	Alg. 3.1 ($\theta = 9$)	Alg. 3.1 ($\theta = 12$)	Alg. 3.1 ($\theta = 15$)
p=5	CPU	0.0819	0.0478	0.3300	0.0356	0.0318	0.0323	0.0328	0.0285
	Iter.	9	14	48	5	5	5	5	5
p=10	CPU	0.0870	0.0462	0.4494	0.0501	0.0481	0.0603	0.0530	0.0519
	Iter.	10	11	56	6	6	6	6	6
p=15	CPU	0.1107	0.0513	0.5404	0.0574	0.0541	0.0527	0.0572	0.0578
	Iter.	11	12	61	6	6	6	6	6
p=20	CPU	0.1138	0.0608	0.4254	0.0723	0.0554	0.0575	0.0587	0.0738
	Iter.	11	12	65	7	7	7	7	7

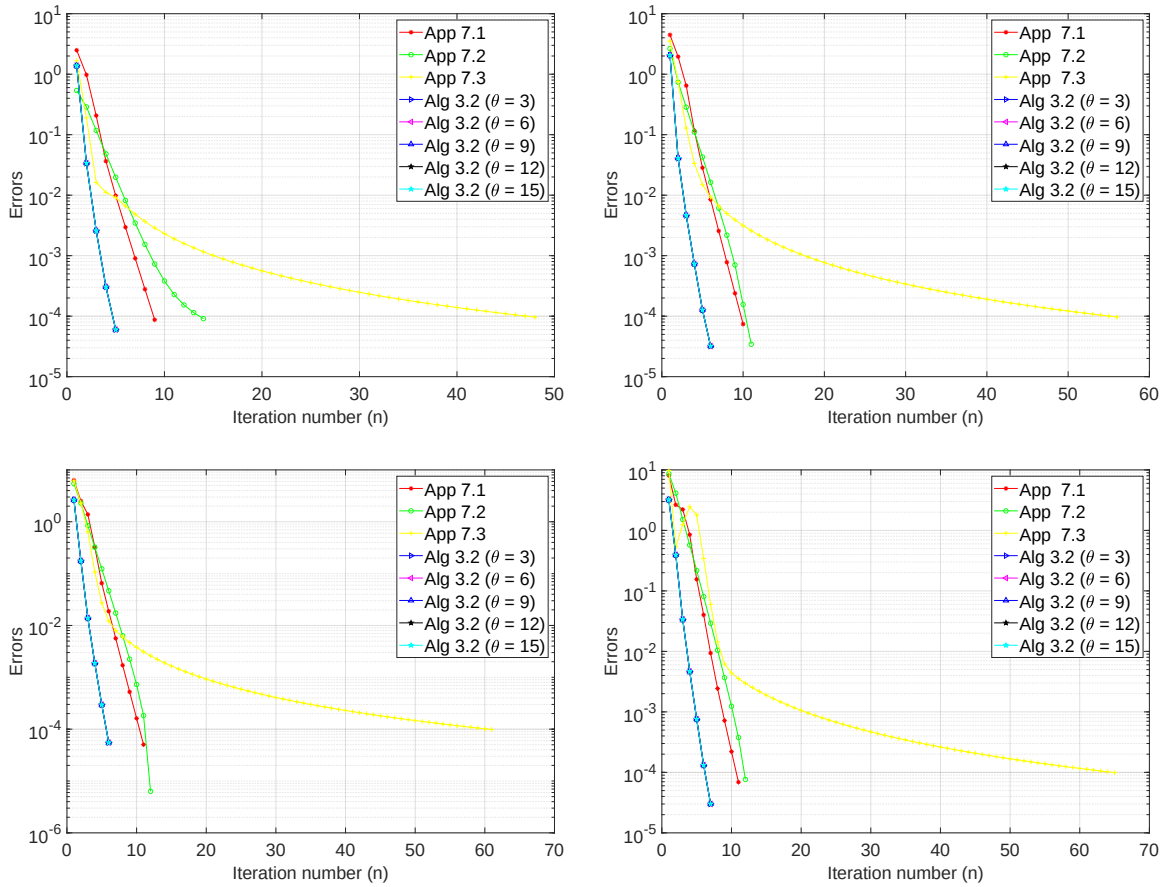

FIGURE 1. Top left: p = 5; Top right: p = 10; Bottom left: p = 15; Bottom right: p = 20.

Table 2. Numerical results for Example 5.2 (Experiment 5.2).

Cases		App. 7.1	App. 7.2	App. 7.3	Alg. 3.1 ($\varepsilon_n = \frac{1}{(2n+1)^3}$)	Alg. 3.1 ($\varepsilon_n = \frac{2}{(2n+5)^3}$)	Alg. 3.1 ($\varepsilon_n = \frac{1}{(2n+1)^4}$)	Alg. 3.1 ($\varepsilon_n = \frac{2}{(2n+5)^4}$)	Alg. 3.1 ($\varepsilon_n = \frac{1}{(2n+1)^5}$)
	CPU	0.1510	0.1280	0.6744	0.0838	0.0852	0.0930	0.0777	0.0803
p=5	Iter.	10	22	47	7	7	7	7	7
	CPU	0.0949	0.0786	0.4007	0.0781	0.0620	0.0580	0.0589	0.0739
p=10	Iter.	12	23	57	8	8	8	8	8
	CPU	0.1115	0.0480	0.4606	0.0782	0.0848	0.0754	0.0803	0.0934
p=20	Iter.	13	11	69	9	9	9	9	9
	CPU	0.0892	0.0803	0.5478	0.0551	0.0723	0.0804	0.0767	0.0783
p=30	Iter.	13	13	77	9	9	9	9	9

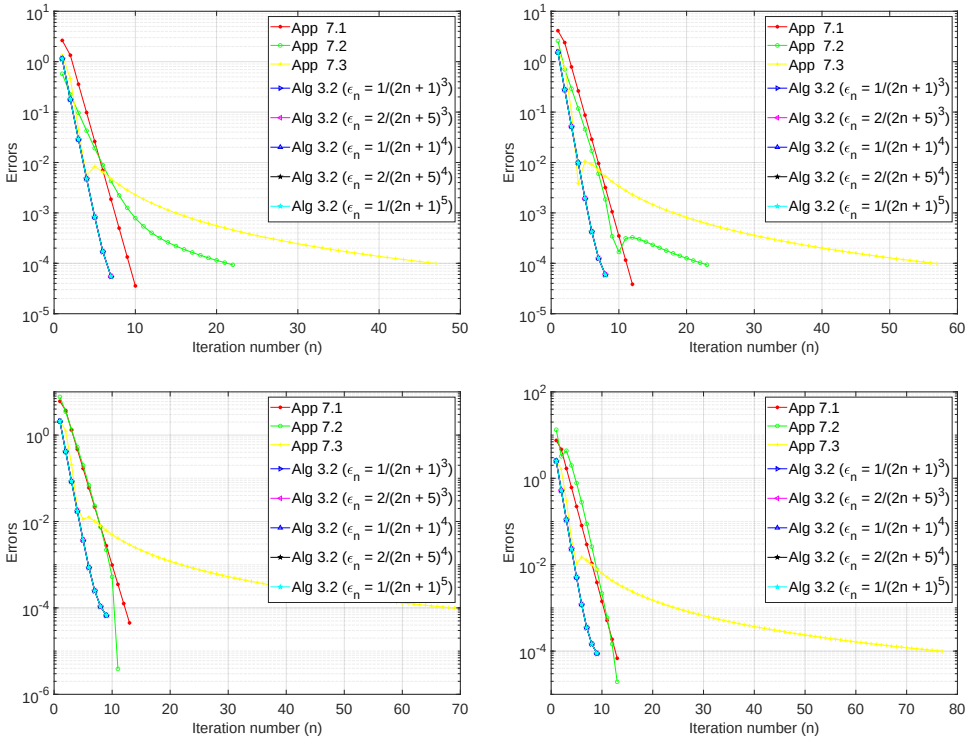

FIGURE 2. Top left: p = 5; Top right: p = 10; Bottom left: p = 20; Bottom right: p = 30.

Table 3. Numerical results for Example 5.3 (Experiment 5.3).

Cases		App. 7.1	App. 7.2	App. 7.3	Alg. 3.1 ($\rho = 0.5$)	Alg. 3.1 ($\rho = 1.0$)	Alg. 3.1 ($\rho = 1.5$)	Alg. 3.1 ($\rho = 2.0$)	Alg. 3.1 ($\rho = 2.5$)
1	CPU	0.0054	0.0047	0.0044	0.0152	0.0113	0.0113	0.0121	0.0107
	Iter.	24	24	44	9	9	9	9	9
2	CPU	0.0037	0.0067	0.0041	0.0115	0.0098	0.0099	0.0101	0.0098
	Iter.	20	24	44	9	9	9	9	9
3	CPU	0.0008	0.0018	0.0012	0.0029	0.00251	0.0031	0.0025	0.0031
	Iter.	24	24	44	9	9	9	9	9
4	CPU	0.0026	0.0036	0.0027	0.0027	0.0031	0.0044	0.0027	0.0023
	Iter.	20	24	44	9	9	9	9	9

Remark 5.1. We use distinct starting points, values of p and vary the other parameters in Experiment 1, Experiment 2 and Experiment 3, respectively, we obtain the numerical results shown in Tables 1-3 and Figures 1-3. We compared our method, Algorithm 3.1 with the methods in Appendix 7.1, Appendix 7.2, and Appendix 7.3. In addition, we observe the following from our numerical tests and experiments:

- In the numerical experiments, we randomly selected the parameters and noted that regardless of the choices made, the number of iteration does not change and no significant difference in the CPU time.

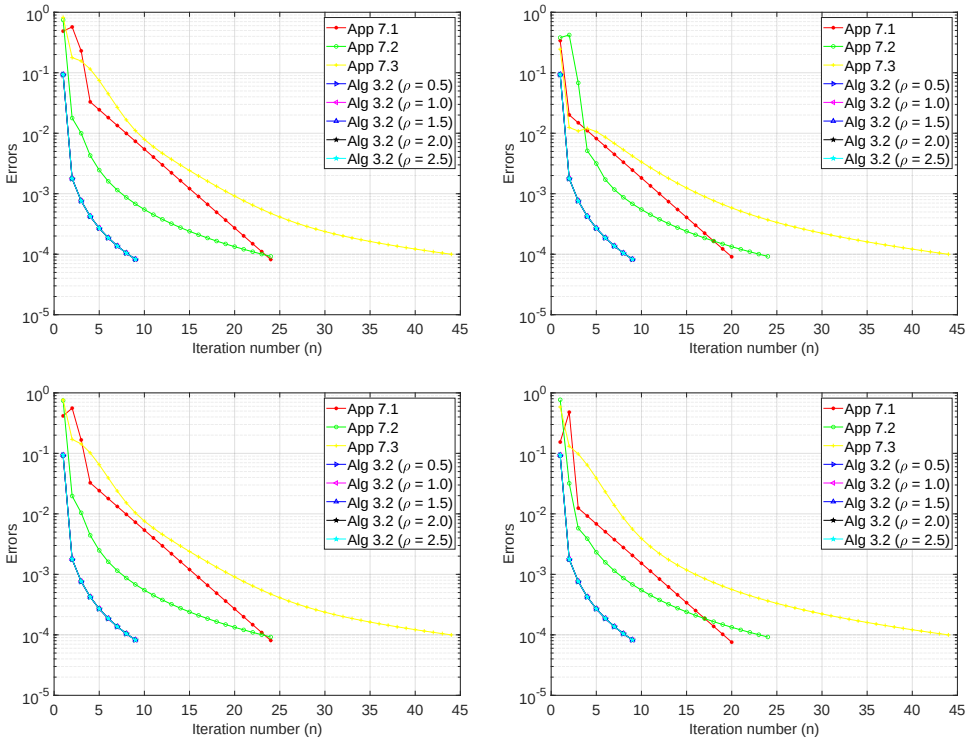

FIGURE 3. Top left: Case 1; Top right: Case 2; Bottom left: Case 3; Bottom right: Case 4.

- In Experiment 1, we examine the sensitivity of θ for each case of p in order to know if the choices of θ affect the performance of our method. Clearly, from Table 1 and Fig. 1, the number of iterations for our method is well behaved for $\theta \in \{3.0, 6.0, 9.0, 12.0, 15.0\}$. In addition, there is no significant difference in the CPU time as we vary the value of θ.
- In Experiment 2, we examine the sensitivity of ε_n for each case of p in order to know if the choices of ε_n affect the performance of our method. Clearly, from Table 2 and Fig. 2, the number of iterations for our method is well behaved for

$$\varepsilon_n \in \{\frac{1}{(2n+1)^3}, \frac{2}{(2n+5)^3}, \frac{1}{(2n+1)^4}, \frac{2}{(2n+5)^4}, \frac{1}{(2n+1)^5}\}.$$

  Moreover, there is no clear difference in the CPU time as we vary the value of $\beta_{n,i}$.
- In Experiment 3, we examine the sensitivity of ρ for each starting points in order to know if the choices of ρ affect the performance of our method. Clearly, from Table 3 and Fig. 3, the number of iterations for our method is well behaved for $\rho \in \{0.5, 1.0, 1.5, 2.0, 2.5\}$. In addition, there is no significant difference in the CPU time as we vary the value of θ.
- From Tables 1-3, Fig. 1, Fig. 2 and Fig. 3, we noted clearly that in terms of number of iterations, our method, Algorithm 3.1 performs better than the existing methods in Appendix 7.1, Appendix 7.2 and Appendix 7.3 and no significant difference in the CPU time.

6. Conclusion

In this paper, we investigated a common solution to a pseudo-monotone equilibrium and a fixed point problem of Bregman quasi-nonexpansive mappings in a p-uniformly convex Banach space by using an inertial Halpern subgradient extragradient iteration. The dependence on prior knowledge of the Lipschitz constants for the pseudo-monotone operator was eliminated by using a self-adaptive step size. We demonstrated that our approach strongly converges to a common element in the solution set.

Acknowledgments
The research of the first author was supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant No. 119903). The second author was supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant No. 119903). The third author acknowledges with thanks the bursary and financial support from University of KwaZulu Natal, Postdoctoral Fellowship. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to funding organization.

References

[1] P.N. Anh, A hybrid extragradient method extended to fixed point problems and equilibrium problems, Optimization 2 (2013), 271–283.
[2] A. Beck, M. Teboulle, A fast iterative shrinkage-thresholding algorithm for linear inverse problems, SIAM J. Imaging Sci. 2 (2009), 183-202.
[3] L. M. Bregman, The relaxation method for finding the common point of convex sets and its application to solution of problems in convex programming, U.S.S.R Comput. Math. Phys. 7 (1967), 200-217.
[4] C. E. Chidume, Geometric Properties of Banach Spaces and Nonlinear Iterations, Springer, 2009. DOI: https://doi.org/10.1007/978-1-84882-190-3
[5] S.Y. Cho, A monotone Bregam projection algorithm for fixed point and equilibrium problems in a reflexive Banach space, Filomat, 34 (2020), 1487–1497.
[6] S.Y. Cho, A convergence theorem for generalized mixed equilibrium problems and multivalued asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 21 (2020), 1017-1026.
[7] G. Z. Eskandani, M. Raeisi, T.M. Rassias, A hybrid extragradient method for solving pseudomonotone equilibrium problems using Bregman distance, J. Fixed Point Theory Appl. 20 (2018), 132.
[8] J. Fan, L. Liu, X. Qin, A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities, Optimization, 69 (2020), 2199-2215.
[9] K. Fan, A minimax inequality and applications, In: Shisha, O. (ed.) Inequalities, vol. III, pp 103-113, Academic Press, New York, 1972.
[10] E.C. Godwin, O.T., Mewomo, T.O. Alakoya, A strongly convergent algorithm for solving multiple set split equality equilibrium and fixed point problems in Banach spaces, Proc. Edinb. Math. Soc. 66 (2023), 575-515.

[11] E.C. Godwin, T.O. Alakoya, O.T. Mewomo, J.-C. Yao, Relaxed inertial Tseng extragradient method for variational inequality and fixed point problems, Appl. Anal. 102 (2023), 4253-4278.
[12] D.V. Hieu, Halpern subgradient extragradient method applied to equilibrium problems, RASCAM, 111 (2017), 823-840.
[13] D.V. Hieu, J.J. Strodiot, Strong convergence theorems for equilibrium problems and fixed point problems in Banach spaces, J Fixed Point Theory Appl. 20 (2018), 131.
[14] L.O. Jolaoso, O.K. Oyewole, K.O. Aremu, O.T. Mewomo, A new efficient algorithm for finding common fixed points of multivalued demicontractive mappings and solutions of split generalized equilibrium problems in Hilbert spaces. Int. J. Comput. Math. 98 (2021), 1892–1919.
[15] Y. Kimura, S. Saejung, Strong convergence for a common fixed points of two different generalizations of cutter operators, Linear Nonlinear Anal. 1 (2015), 53-65.
[16] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Ekon. Matem. Metody 12 (1976), 747–756.
[17] L. W. Kuo, D. R. Sahu, Bregman distance and strong convergence of proximal-type algorithms, Abstr. Appl. Anal. 2013 (2013), 590519.
[18] L. Liu, S.Y. Cho, J.C. Yao, Convergence analysis of an inertial Tseng's extragradient algorithm for solving pseudomonotone variational inequalities and applications, J. Nonlinear Var. Anal. 5 (2021), 627-644.
[19] O.T. Mewomo, R.N. Nwokoye, C.C. Okeke, Two-step inertial Tseng's extragradient method for solving quasimonotone variational inequalities, Quaest. Math. 47 (2024), 1505–1543.
[20] O.T. Mewomo, V.A. Uzor, A. Gibali, An alternated inertial projection and contraction algorithm for solving quasimonotone bilevel variational inequalities with application to optimal control problems, Acta. Appl. Math. 193 (2024), 1.
[21] K. Muangchoo, A new strongly convergent algorithm to solve pseudo-monotone equilibrium problems in a real Hilbert space, J. Math, Computer Sci. 24 (2022), 308-322.
[22] R. Maluleka, G.C. Ugwunnadi, C.C. Okeke, M. Aphane, On pseudomonotone equilibrium and fixed point problems in Hadamard spaces with applications, J. Nonlinear Funct. Anal. 2024 (2024), 7.
[23] E. Naraghirad, J.C. Yao, Bregman weak relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. 2013 (2013), 141.
[24] B.T. Polyak, Some methods of speeding up the convergence of iterates methods, U.S.S.R Comput. Math. Phys. 4 (1964), 1-17.
[25] J.S. Pang, F. Facchinei, Finite-dimensional Variational Inequalities and Complementarity Problems, Springer, New York, 2007.
[26] I.M. Proudnikov, Nash equilibrium points and their finding for nonsmooth case, Commun. Optim. Theory 2024 (2024) 16.
[27] T.D. Quoc, L.D. Muu, V.H. Nguyen, Extragradient algorithms extended to equilibrium problems, Optimization 57 (2008), 749–776.
[28] X. Qin, A weakly convergent method for splitting problems with nonexnpansive mappings, J. Nonlinear Convex Anal. 24 (2023), 1033-1043.

[29] S. Reich, S. Sabach, A strong convergence theorem for a proximal-type algorithm in reflexive Banach space, J. Nonlinear Convex Anal. 10 (2009), 471-485.
[30] M. Raeisi, G.Z. Eskandani, A hybrid extragradient method for a general split equality problem involving resolvents and pseudomonotone bifunctions in Banach spaces, Calcolo, 56 (2019), 43.
[31] Y. Shehu, F. U. Ogbuisi, O. S. Iyiola, Convergence analysis of an iterative algorithm for fixed point problems and split feasibility problems in certain Banach spaces, Optimization, 65 (2016), 299-323.
[32] Y. Shehu, J.C. Yao, Weak convergence of two-step inertial iteration for countable family of quasi-nonexpansive mappings, Ann. Math. Sci. Appl. 7 (2022), 259-279.
[33] Y. Shehu, et al., Strongly convergent inertial extragradient type methods for equilibrium problems, Appl. Anal. 102 (2023), 2160-2188.
[34] A. Taiwo, O.T. Mewomo, Inertial viscosity with alternative regularization for certain optimization and fixed point problems, J. Appl. Numer. Optim. 4 (2022), 405-423.
[35] J.V. Tiel, Convex Analysis: An introductory Text, Wiley, New York, 1984.
[36] B. Tan, X. Qin, An alternated inertial algorithm with weak and linear convergence for solving monotone inclusions, Ann. Math. Sci. Appl. 8 (2023), 321-345.
[37] B. Tan, S.Y. Cho, J.C. Yao, Accelerated inertial subgradient extragradient algorithms with non-monotonic step sizes for equilibrium problems and fixed point problems, J. Nonlinear Var. Anal. 6 (2022), 89-122.
[38] V.A. Uzor, T.O. Alakoya, O.T. Mewomo, Modified forward-backward splitting method for split equilibrium, variational inclusion, and fixed point problems, Appl. Set-Valued Anal. Optim. 5 (2023), 95-119.
[39] Z. B. Xu, G. F. Roach, Characteristics inequalities of uniformly convex and uniformly smooth Banach spaces, J. Math. Anal. Appl. 157 (1991), 189-210.
[40] J. Yang, H. Liu, The subgradient extragradient method extended to pseudomonotone equilibrium problems and fixed point problems in Hilbert space, Optim. Lett. 14 (2020), 1803-1816.
[41] H. Zhou, X. Qin, Fixed Points of Nonlinear Operators, De Gruyter STEM, Berlin, 2020.

7. APPENDIX

Algorithm 7.1. [13]

Initialization: Choose $x_0,\ u \in C$ and the sequences $\{\alpha_n\},\{\beta_n\}$ and $\{\lambda_n\}$ such that

$$
\begin{aligned}
y_n &= \arg\min\{\lambda_n f(x_n,y) + \frac{1}{2}\phi(y,x_n) : y \in C\},\\
z_n &= \arg\min\{\lambda_n f(y_n,y) + \frac{1}{2}\phi(y,x_n) : y \in C\}\\
x_{n+1} &= \Pi_C J^{-1}\left(\gamma_n J(u) + (1-\gamma_n)(\beta_n J z_n + (1-\beta_n) JDz_n)\right),
\end{aligned}
$$

where $0 < a \leq \lambda_n \leq b < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}, D : E \to E$ is a quasi-ϕ-nonexpansive mapping.

Algorithm 7.2. [13]

Initialization: Choose $x_0,\ u \in C$ and the sequences $\{\alpha_n\},\ \{\beta_n\}$ and $\{\lambda_n\}$ such that

$$
\begin{aligned}
&y_n = \arg\min\{\lambda_n f(x_n,y) + \frac{1}{2}\phi(y,x_n) : y \in C\},\\
&\text{Find } m \text{ the smallest nonnegative integer such that}\\
&\begin{cases} z_{m,n} = (1-\xi^m)x_n + \xi^m y_n,\\ f(z_{m,n},x_n) - f(z_{m,n},y_n) \geq \frac{\alpha}{2\lambda_n}\phi(y_n,x_n). \end{cases}\\
&\text{Set } \rho_n = \xi^m \text{ and } z_n = z_{m,n}\\
&\text{choose } g_n \in \partial_2 f(z_n,x_n) \text{ and compute } w_n = \Pi_C J^{-1}(Jx_n - \sigma_n g_n)\\
&\text{where } \sigma_n = \frac{v f(z_n,x_n)}{\|g_n\|^2} \text{ if } y_n \neq x_n \text{ and } \sigma_n = 0 \text{ otherwise.}\\
&x_{n+1} = \Pi_C J^{-1}\left(\gamma_n J(u) + (1-\gamma_n)(\beta_n J w_n + (1-\beta_n) JDw_n)\right),
\end{aligned}
$$

where $\{\lambda_n\} \subset [\lambda,1], \lambda \in (0,1], D : E \to E$ is a quasi-ϕ-nonexpansive mapping.

Algorithm 7.3. [7]

Initialization: Iterative Step: Suppose $\{x_n\}$ is a sequece generated by $x_1 \in C$. Calculate x_{n+1} as follows:

$$
\begin{aligned}
w_n^i &= \arg\min\{\lambda_n g_i(x_n,w) + D_f(w,x_n) : w \in C\}\ i = 1,...,N,\\
z_n^i &= \arg\min\{\lambda_n g_i(w_n^i,z) + D_f(z,x_n) : z \in C\}\ i = 1,...,N,\\
i_n &\in \arg\max\{D_f(z_n^i,x_n),\ i = 1,2,...,N\},\ \bar{z}_n = z_n^{i_n},\\
y_n &= \nabla f^*\left(\eta_{n,0}\nabla f(\bar{z}_n) + \sum_{j=1}^{m}\eta_{n,j}\nabla f(z_{n,j})\right),\ z_{n,j} \in T_j\bar{z}_n,\\
x_{n+1} &= \mathrm{Proj}_C^f(\nabla f^*(\gamma_n \nabla f(q_n)) + (1-\gamma_n)\nabla f(y_n)))
\end{aligned}
$$

where $T_j : C \to CB(C)$ is a multivalued Bregman relatively nonexpansive mapping, $\{\lambda_n\} \subset [a,b] \subset (0,p)$, where $p = \min\{\frac{1}{c_1}, \frac{1}{c_2}\}$, $\lim_{n\to\infty} q_n = q \in E$.