Appl. Set-Valued Anal. Optim. 8 (2026), No. 2, pp. 205-222
Available online at http://asvao.biemdas.com
https://doi.org/10.23952/asvao.8.2026.2.05

IMPROVED JENSEN'S DRAGOMIR TYPE INEQUALITY FOR (ϕ,h)-CONVEX FUNCTIONS

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Abstract. In this paper, we present several new inequalities for (ϕ,h)-convex functions that complement the existing inequalities in this field. Additionally, we examine multiple-term refinements of the celebrated Jensen-type inequality for (ϕ,h)-convex functions. We also extend some results from the literature by using the theory of sub-majorization.

Keywords. (ϕ,h)-convex function; Jensen's inequality; Weak sub-majorization.
2020 Mathematics Subject Classification. 26D07, 47A63.

1. INTRODUCTION AND PRELIMINARIES

Convex functions are crucial in numerous mathematical fields, such as nonlinear analysis, convex and nonconvex optimization, mathematical physics, and operator theory. Recall that a convex function $f: I \to \mathbb{R}$ is a function that satisfies

$$f(\eta u+\theta v) \leq \eta f(u)+\theta f(v), \tag{1.1}$$

for every $u,v \in I$ and $\eta,\theta > 0$ such that $\eta+\theta=1$ and f is said to be log-convex if f is positive and $\log f$ is convex. A current research trend in mathematical inequalities involves minimizing the difference between the two sides of (1.1) by introducing specific terms. This inequality has been refined in the literature and has numerous applications for both scalars and matrices.

In this paper, we use I and $[a,b]$ to denote a ϕ-convex subset of $\mathbb{R}$, which ϕ is a continuous function that is strictly monotone on an interval I. A subset $I \subset \mathbb{R}$ is called ϕ-convex if, for all $u,v \in I$ and $\eta,\theta \in (0,1)$ with $\eta+\theta=1$, the element $\phi^{-1}[\eta\phi(u)+\theta\phi(v)]$ belongs to I. Let $h: J \to \mathbb{R}$ be a non-negative, non-zero function with J as a subset of $\mathbb{R}$, and let $f: I \to \mathbb{R}$ be a function. Recall that f is called (ϕ,h)-convex if I is a ϕ-convex set and, for every $u,v \in I$, and

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Received 4 December 2024; Accepted 30 October 2025; Published online 17 January 2026.

$\eta, \theta > 0$ such that $\eta + \theta = 1$, the following inequality holds

$$f\left(\phi^{-1}\left[\eta\phi(u) + \theta\phi(v)\right]\right) \leq h(\eta)f(u) + h(\theta)f(v), \tag{1.2}$$

with the notations previously defined. We recall that f is called a (ϕ, h) log-convex function if it satisfies the following inequality

$$f\left(\phi^{-1}\left[\eta\phi(u) + \theta\phi(v)\right]\right) \leq f^{h(\eta)}(u)f^{h(\theta)}(v).$$

Clearly, if $h = \mathrm{id}$ and $\phi = \mathrm{id}$ (where id denotes the identity function) in (1.2), we obtain the classical definition of convexity. Additionally, if $\phi(x) = x^p$ for $p \in \mathbb{R}$, we see the celebrated definition of (p,h)-convexity [4].

The celebrated Jensen inequality for convex functions generalizes (1.1) to n parameters as follows

$$f\left(\sum_{i=1}^{n} \eta_i u_i\right) \leq \sum_{i=1}^{n} \eta_i f(u_i), \tag{1.3}$$

where $f : I \to \mathbb{R}$ is a convex function, $\{u_1, \ldots, u_n\} \subset I$ and $\{\eta_1, \ldots, \eta_n\} \subset [0,1]$ are such that $\sum_{i=1}^{n} \eta_i = 1$. By applying Jensen's inequality (1.3) to $\log f$, we derive the following Jensen type inequality associated with log-convexity

$$f\left(\sum_{i=1}^{n} \eta u_i\right) \leq \prod_{i=1}^{n} f^{\eta_i}(u_i).$$

Here f is log-convex.

From now on, we examine the following functional associated with the Jensen's inequality

$$\mathscr{J}(f, \mathbf{u}, \eta) = \sum_{i=1}^{n} \eta_i f(u_i) - f\left(\sum_{i=1}^{n} \eta_i u_i\right).$$

For $\eta = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$, we define $\mathscr{J}(f, \mathbf{u})$ by $\mathscr{J}_n(f, \mathbf{u}) = \sum_{i=1}^{n} \frac{1}{n} f(u_i) - f\left(\sum_{i=1}^{n} \frac{1}{n} u_i\right)$. In [12], a refinement and reversed of (1.3) was introduced and demonstrated as follows

$$n\eta_{\min} \mathscr{J}_n(f, \mathbf{u}) \leq \mathscr{J}(f, \mathbf{u}, \eta) \leq n\eta_{\max} \mathscr{J}_n(f, \mathbf{u}), \tag{1.4}$$

where $\eta_{\min} = \min\{\eta_1, \ldots, \eta_n\}$ and $\eta_{\max} = \max\{\eta_1, \ldots, \eta_n\}$. Dragomir provided another generalization of inequality (1.4), which reads

$$m\mathscr{J}(f, \mathbf{u}, \theta) \leq \mathscr{J}(f, \mathbf{u}, \eta) \leq M\mathscr{J}(f, \mathbf{u}, \theta), \tag{1.5}$$

where $m = \min_{1 \leq j \leq n}\left\{\frac{\eta_j}{\theta_j}\right\}$ and $M = \max_{1 \leq j \leq n}\left\{\frac{\eta_j}{\theta_j}\right\}$. Very recently, The, Van and Huy [23] demonstrated a significant improvement of inequalities (1.5), which reads

$$m\mathscr{J}(f, \mathbf{u}, \theta) + \mathbf{m}(|J|+1)\mathscr{H}_J \leq \mathscr{J}(f, \mathbf{u}, \eta) \leq m\mathscr{J}(f, \mathbf{u}, \theta) + \mathscr{M}(|J|+1)\mathscr{H}_J, \tag{1.6}$$

where $J = \{i : \ \eta_i - m\theta_i \neq 0\}$, $|J|$ is the cardinal of J, $\mathbf{m} = \min_{i \in J}\{m, \eta_i - m\theta_i\}$, $\mathscr{M} = \max_{i \in J}\{m, \eta_i - m\theta_i\}$, and

$$H_J := \frac{1}{|J|+1}\left[\sum_{i \in J} f(u_i) + f\left(\sum_{i=1}^{n} \theta_i u_i\right)\right] - f\left(\frac{1}{|J|+1}\left(\sum_{i \in J} u_i + \sum_{i=1}^{n} \theta_i u_i\right)\right).$$

Readers are encouraged to refer to [3, 20] for further exploration of recent advancements related to Jensen's inequality.

The functional related to Jensen's inequality for h-convex functions is

$$\mathscr{J}_h(f,\mathbf{u},\eta)=\sum_{i=1}^{n}h(\eta_i)f(u_i)-f\left(\sum_{i=1}^{n}\eta_i u_i\right),$$

where $\mathbf{u}=(u_1,u_2,\ldots,u_n)\in I^n$ and $f:I\to\mathbb{R}$ is a h-convex function. The corresponding non-weighted functional, i.e., when $\eta=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$, is

$$\mathscr{J}_h(f,\mathbf{u})=h\left(\frac{1}{n}\right)\sum_{i=1}^{n}f(u_i)-f\left(\frac{1}{n}\sum_{i=1}^{n}u_i\right).$$

Under the assumption that f is h-convex, it is evident that $\mathscr{J}_h(f,\mathbf{u},\eta)$ and $\mathscr{J}_h(f,\mathbf{u})$ are non-negative.

A recent and significant extension of inequality (1.5) for h-convex functions was established by Krnic et al. [9] as follows. Given a function $f:I\to\mathbb{R}$, an h-convex function and $h:J\to\mathbb{R}$, a superadditive and supermultiplicative function (see Section 2), the inequalities

$$h(m)\mathscr{J}_h(f,\mathbf{u},\theta)\leq\mathscr{J}_h(f,\mathbf{u},\eta)\leq\frac{1}{h\left(\frac{1}{M}\right)}\mathscr{J}_h(f,\mathbf{u},\theta) \tag{1.7}$$

hold for all $\mathbf{u}\in I^n$, where $m=\min_{1\leq i\leq n}\left\{\frac{\eta_i}{\theta_i}\right\},M=\max_{1\leq i\leq n}\left\{\frac{\eta_i}{\theta_i}\right\}$ and $h\left(\frac{1}{M}\right)\neq 0$. For more extensive generalizations to a wider class of functions, such as (p,h)-convexity, readers are advised to consult [7]. The recognized Jensen-type inequality for (ϕ,h)-convexity (see [22]), where h is a non-negative super-multiplicative function, reads

$$f\left(\phi^{-1}\left[\sum_{i=1}^{n}\eta_i\phi(u_i)\right]\right)\leq\sum_{i=1}^{n}h(\eta_i)f(u_i). \tag{1.8}$$

Using Jensen's inequality on (ϕ,h)-convex function $\log f$ results in the following inequality

$$f\left(\phi^{-1}\left[\sum_{i=1}^{n}\eta_i\phi(u_i)\right]\right)\leq\prod_{i=1}^{n}f^{h(\eta_i)}(u_i). \tag{1.9}$$

Consider the functional below, which is related to Jensen's inequality for (ϕ,h)-convex functions

$$\mathscr{J}_{(\phi,h)}(f,\mathbf{u},\eta)=\sum_{i=1}^{n}h(\eta_i)f(u_i)-f\left(\phi^{-1}\left[\sum_{i=1}^{n}\eta_i\phi(u_i)\right]\right).$$

The corresponding non-weighted functional, i.e., when $\eta=\left(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right)$, is defined by

$$\mathscr{J}_{(\phi,h)}(f,\mathbf{u})=\sum_{i=1}^{n}h\left(\frac{1}{n}\right)f(u_i)-f\left(\phi^{-1}\left[\sum_{i=1}^{n}\frac{1}{n}\phi(u_i)\right]\right).$$

Naturally, the Jensen-type inequality for (ϕ,h)-convex functions leads to the positivity of functionals $\mathscr{J}_{(\phi,h)}(f,\mathbf{u},\eta)$ and $\mathscr{J}_{(\phi,h)}(f,\mathbf{u})$.

In this paper, we aim to broaden the scope of the inequalities to include a more general class related to (ϕ,h)-convexity. This work builds on and complements the existing results in the literature concerning convex, log-convex, (p,h)-convex, and (p,h)-log-convex functions.

The paper is structured as follows: In Section 2, we extend the inequalities discussed in the introduction to the class of (ϕ,h)-convex functions, which generalize a wide range of inequalities related to convexity and (p,h)-convexity found in the literature. In Section 3, we further generalize the results obtained in Section 2 through the theory of weak submajoration. Section 4 ends this paper with concluding remarks.

2. PRELIMINARIES AND AUXILIARY RESULTS

We begin this section by recalling several classes of functions. Recall that a function $h: J \to \mathbb{R}$ is said to be super-multiplicative if, for all $u, v \in J$, $uv \in J$ and

$$h(u)h(v) \leq h(uv). \tag{2.1}$$

If inequality (2.1) is reversed, h is referred to as a sub-multiplicative function. If equality holds in (2.1), h is called a multiplicative function. Besides, if $u+v \in J$ and

$$h(u)+h(v) \leq h(u+v), \tag{2.2}$$

then h is said to be a super-additive function. If inequality (2.2) is reversed, we say that h is a sub-additive function. If equality (2.2) holds, we say that h is an additive function. Concrete examples of functions that are superadditive and super-multiplicative can be found in [7]. In this paper, h is assumed to be super-multiplicative and super-additive.

We are now prepared to demonstrate our first main result. We would like to highlight that this result refines inequality (1.8), thereby enabling us to establish the general form of the first inequality in (1.5), which relates to inequality (1.5) for (ϕ,h)-convex functions.

Theorem 2.1. *Let f be a positive (ϕ,h)-convex function on $[a,b]$, and let $\{u_1,\ldots,u_n\} \subset [a,b]$ and $\{\eta_1,\ldots,\eta_n,\theta_1,\ldots,\theta_n\} \subset (0,1)$ be such that $\sum_{i=1}^n \eta_i = \sum_{i=1}^n \theta_i = 1$. Then $\mathscr{J}_{(\phi,h)}(f,\boldsymbol{u},\eta) \geq h(m)\,\mathscr{J}_{(\phi,h)}(f,\boldsymbol{u},\theta)$, where $m = \min_{1\leq j\leq n}\left\{\frac{\eta_j}{\theta_j}\right\}$.*

Proof. Since h is both super-multiplicative and super-additive, we have

$$\begin{aligned}
\mathscr{I} &:= \sum_{i=1}^n h(\eta_i) f(u_i) - h(m)\,\mathscr{J}_{(\phi,h)}(f,\mathbf{u},\theta) \\
&= \sum_{i=1}^n \big(h(\eta_i) - h(m)h(\theta_i)\big) f(u_i) + h(m) f\left(\phi^{-1}\left[\sum_{i=1}^n \theta_i \phi(u_i)\right]\right) \\
&\geq \sum_{i=1}^n \big(h(\eta_i) - h(m\theta_i)\big) f(u_i) + h(m) f\left(\phi^{-1}\left[\sum_{i=1}^n \theta_i \phi(u_i)\right]\right) \\
&\geq \sum_{i=1}^n \left[h(\eta_i - m\theta_i)\right] f(u_i) + h(m) f\left(\phi^{-1}\left[\sum_{i=1}^n \theta_i \phi(u_i)\right]\right).
\end{aligned}$$

Observe that $\eta_i - m\theta_i \geq \eta_i - \frac{\eta_i}{\theta_i}\theta_i = 0$ and $\sum_{i=1}^n (\eta_i - m\theta_i) + m = 1$. By applying the definition of (ϕ,h)-convexity, we obtain that

$$\mathscr{I} \geq f\left(\phi^{-1}\left[\sum_{i=1}^n \left[(\eta_i - m\theta_i)\,\phi(u_i) + m\sum_{i=1}^n \theta_i\phi(u_i)\right]\right]\right).$$

which is precisely equal to $f\left(\phi^{-1}\left[\sum_{i=1}^n \eta_i\phi(u_i)\right]\right)$. This concludes the proof. □

Remark 2.1. Theorem 2.1 is a natural extension of several celebrated results from the literature.

1. If we choose $\phi(t) = t^p$ for $p \in \mathbb{R}$ in Theorem 2.1, we retrieve [7, Theorem 2.2].
2. If we set $\phi(t) = t$ in Theorem 2.1, then we recover [9, Theorem 2].

As a consequence of the previous theorem, we derive the following multiplicative refinement of the Jensen-type inequality for (ϕ,h) log-convex functions, which extends [1, Lemma 2.1] to the concept of (ϕ,h) log-convexity.

Corollary 2.1. *Let f be a positive (ϕ,h) log-convex function on $[a,b]$, and let $\{u_1,\dots,u_n\} \subset [a,b]$, and $\{\eta_1,\dots,\eta_n,\theta_1,\dots,\theta_n\} \subset (0,1)$ be such that $\sum_{i=1}^n \eta_i = \sum_{i=1}^n \theta_i = 1$. Then,*

$$\left(\frac{f\left(\phi^{-1}\left[\sum_{i=1}^n \theta_i\phi(u_i)\right]\right)}{\prod_{i=1}^n f^{h(\theta_i)}(u_i)}\right)^{h(m)} f\left(\phi^{-1}\left[\sum_{i=1}^n \eta_i\phi(u_i)\right]\right) \leq \prod_{i=1}^n f^{h(\eta_i)}(u_i),$$

where $m = \min\limits_{1\leq j\leq n}\left\{\frac{\eta_j}{\theta_j}\right\}$.

Now, let us introduce the reverse of the previous theorem, which generalizes the second inequality in (1.7) to the framework of (ϕ,h)-convexity.

Theorem 2.2. *Let f be a positive (ϕ,h)-convex function on $[a,b]$, and let $\{u_1,\dots,u_n\} \subset [a,b]$ and $\{\eta_1,\dots,\eta_n,\theta_1,\dots,\theta_n\} \subset (0,1)$ be such that $\sum_{i=1}^n \eta_i = \sum_{i=1}^n \theta_i = 1$. Then*

$$\mathscr{J}_{(\phi,h)}(f,\boldsymbol{u},\eta) \leq \frac{1}{h\left(\frac{1}{M}\right)}\mathscr{J}_{(\phi,h)}(f,\boldsymbol{u},\theta),$$

where $M = \max_{1\leq j\leq n}\left\{\frac{\eta_j}{\theta_j}\right\}$.

Proof. Assume that h is both multiplicative and super-additive. Then

$$\begin{aligned}\mathscr{I} &:= \sum_{i=1}^n h(\theta_i)f(u_i) - h\left(\frac{1}{M}\right)\sum_{i=1}^n h(\eta_i)f(u_i) + h\left(\frac{1}{M}\right)f\left(\phi^{-1}\left[\sum_{i=1}^n \eta_i\phi(u_i)\right]\right)\\ &\geq \sum_{i=1}^n\left(h(\theta_i) - h\left(\frac{\eta_i}{M}\right)\right)f(u_i) + h\left(\frac{1}{M}\right)f\left(\phi^{-1}\left[\sum_{i=1}^n \eta_i\phi(u_i)\right]\right)\\ &\geq \sum_{i=1}^n h\left(\theta_i - \frac{\eta_i}{M}\right)f(u_i) + h\left(\frac{1}{M}\right)f\left(\phi^{-1}\left[\sum_{i=1}^n \eta_i\phi(u_i)\right]\right).\end{aligned}$$

Since $\sum_{i=1}^{n}\left(\theta_i-\frac{\eta_i}{M}\right)+\frac{1}{M}=1$, it can be deduced from the definition of (ϕ,h)-convexity that

$$\mathscr{I}\geq f\left(\phi^{-1}\left[\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right).$$

Thus complete the proof. □

The previous theorem directly leads to the following result for (ϕ,h) log-convex functions, which provides a reversed version of the Jensen-type inequality for (ϕ,h) log-convex functions.

Corollary 2.2. *Let f be a positive (ϕ,h) convex function on $[a,b]$, and let $\{u_1,\dots,u_n\}\subset[a,b]$, $\{\eta_1,\dots,\eta_n,\theta_1,\dots,\theta_n\}\subset(0,1)$ be such that $\sum_{i=1}^{n}\eta_i=\sum_{i=1}^{n}\theta_i=1$. Then*

$$\prod_{i=1}^{n}f^{h(\eta_i)}(u_i)\leq\left(\frac{f\left(\phi^{-1}\left[\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right)}{\prod_{i=1}^{n}f^{h(\theta_i)}(u_i)}\right)^{h\left(\frac{1}{M}\right)}f\left(\phi^{-1}\left[\sum_{i=1}^{n}\eta_i\phi(u_i)\right]\right),$$

where $M=\max_{1\leq j\leq n}\left\{\frac{\eta_j}{\theta_j}\right\}$.

A special case of Theorems 2.1 and 2.2 provides the following extension of inequalities (1.4) to the concept of (ϕ,h)-convexity.

Corollary 2.3. *Let f be a positive (ϕ,h)-convex function on $[a,b]$, and let $\{u_1,\dots,u_n\}\subset[a,b]$ and $\{\eta_1,\dots,\eta_n,\theta_1,\dots,\theta_n\}\subset(0,1)$ be such that $\sum_{i=1}^{n}\eta_i=\sum_{i=1}^{n}\theta_i=1$. Then*

$$h(m)\mathscr{J}_{(\phi,h)}(f,\boldsymbol{u})\leq\mathscr{J}_{(\phi,h)}(f,\boldsymbol{u},\eta)\leq\frac{1}{h\left(\frac{1}{M}\right)}\mathscr{J}_{(\phi,h)}(f,\boldsymbol{u}),$$

where $m=\min_{1\leq j\leq n}\left\{\frac{\eta_j}{\theta_j}\right\}$ *and* $M=\max_{1\leq j\leq n}\left\{\frac{\eta_j}{\theta_j}\right\}$.

The following theorem naturally extends the inequalities in (1.6) from the context of classical convexity to the more general framework of (ϕ,h)-convexity.

Theorem 2.3. *Let f be a positive (ϕ,h)-convex function on $[a,b]$, and let $\{u_1,\dots,u_n\}\subset[a,b]$ and $\{\eta_1,\dots,\eta_n,\theta_1,\dots,\theta_n\}\subset(0,1)$ be such that $\sum_{i=1}^{n}\eta_i=\sum_{i=1}^{n}\theta_i=1$. Then,*

$$\begin{aligned}&h(m)\mathscr{J}_{(\phi,h)}(f,\boldsymbol{u},\theta)+h(\mathbf{m}(|J|+1))\mathscr{H}_J\\&\leq\mathscr{J}_{(\phi,h)}(f,\boldsymbol{u},\eta)\leq h(m)\mathscr{J}_{(\phi,h)}(f,\boldsymbol{u},\theta)+h(\mathscr{M}(|J|+1))\mathscr{H}_J,\end{aligned}$$

where $J=\{i,\ \eta_i-m\theta_i\neq0\}$, $|J|$ is the cardinal of J, $\mathbf{m}=\min_{i\in J}\{m,\eta_i-m\theta_i\}$, $\mathscr{M}=\max_{i\in J}\{m,\eta_i-m\theta_i\}$, and

$$\begin{aligned}\mathscr{H}_J:=h\left(\frac{1}{|J|+1}\right)&\left[\sum_{i\in J}f(u_i)+f\left(\phi^{-1}\left(\sum_{i=1}^{n}\theta_i\phi(u_i)\right)\right)\right]\\&-f\left(\frac{1}{|J|+1}\phi^{-1}\left(\sum_{i\in J}\phi(u_i)+\phi\left(\sum_{i=1}^{n}\theta_iu_i\right)\right)\right).\end{aligned}$$

Proof. Based on the proof of Theorem 2.1, we have

$$\begin{aligned}
\mathscr{I} &:= \sum_{i=1}^{n} h(\eta_i) f(u_i) - h(m)\, \mathscr{J}_{(\phi,h)}(f,\mathbf{u},\theta) \\
&\geq \sum_{i\in J} \left[h(\eta_i - m\theta_i)\right] f(u_i) + h(m) f\left(\phi^{-1}\left[\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right) \\
&\geq f\left(\phi^{-1}\left[\sum_{i\in J}\left[(\eta_i - m\theta_i)\,\phi(u_i) + m\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right]\right) + h((|J|+1)\mathbf{m})\mathscr{H}_J \\
&= f\left(\phi^{-1}\left[\sum_{i=1}^{n}\eta_i\phi(u_i)\right]\right) + h((|J|+1)\mathbf{m})\mathscr{H}_J,
\end{aligned}$$

which indicates the first desired inequality. Similarly, we have

$$\begin{aligned}
\mathscr{I} &\leq f\left(\phi^{-1}\left[\sum_{i\in J}\left[(\eta_i - m\theta_i)\,\phi(u_i) + m\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right]\right) + h((|J|+1)\mathscr{M})\mathscr{H}_J \\
&= f\left(\phi^{-1}\left[\sum_{i=1}^{n}\eta_i\phi(u_i)\right]\right) + h((|J|+1)\mathscr{M})\mathscr{H}_J.
\end{aligned}$$

This completes the proof. □

To address additional related inequalities for (ϕ,h)-convex functions, we need to establish some notations. For the remainder of this section, we denote by $\eta^{(1)} = \left\{\eta_1^{(1)},\ldots,\eta_n^{(1)}\right\} \subset (0,1)$ a convex sequence, satisfying $\sum_{i=1}^{n}\eta_i^{(1)} = 1$. Define $J_1 = \left\{i : \eta_i^{(1)} = \eta_{\min}^{(1)}\right\}$, where $\eta_{\min}^{(1)} = \min\left\{\eta_i^{(1)} : 1 \leq i \leq n\right\}$. The quantity $|J_1|$ stands for the cardinality of J_1. For $k \geq 2$, let $\eta^{(k)}$ be a sequence defined inductively in the following way

$$\eta_i^{(k)} = \begin{cases} \eta_i^{(k-1)} - \eta_{\min}^{(k-1)} & \text{if } \eta_i^{(k-1)} \neq \eta_{\min}^{(k-1)} \\ \frac{1}{J_{k-1}} n\eta_{\min}^{(k-1)} & \text{if } \eta_i^{(k-1)} = \eta_{\min}^{(k-1)} \end{cases} \quad \text{where } J_{k-1} = \left\{i : \eta_i^{(k-1)} = \eta_{\min}^{(k-1)}\right\}, \tag{2.3}$$

and for $k \geq 1, \eta_{\min}^{(k)} = \min\left\{\eta_1^{(k)},\ldots,\eta_n^{(k)}\right\}$. Now, we set $x^{(1)} = \left\{x_1^{(1)},\ldots,x_n^{(1)}\right\} \subset I$, and we provide a new sequence $x^{(k)}$ defined by

$$u_i^{(k)} = \begin{cases} u_i^{(k-1)} & \text{if } \eta_i^{(k-1)} \neq \eta_{\min}^{(k-1)} \\ \phi^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(k-1)})\right] & \text{if } \eta_i^{(k-1)} = \eta_{\min}^{(k-1)} \end{cases}, 1 \leq i \leq n. \tag{2.4}$$

We point out that the order of the $\left\{u_i^{(1)}\right\}$ follows the order in which they are associated with the $\left\{\eta_i^{(1)}\right\}$. That is, $x_1^{(1)}$ is the value multiplied with $\eta_1^{(1)}$, and so on. Bearing those notations in mind, we present the main result of this section, extending Theorem 4.1 from [7] to the notion of (ϕ,h)-convex functions. The proof is similar to the one used in [7].

Theorem 2.4. *Let $f: I \to \mathbb{R}$ be a (ϕ,h)-convex function, and let $\left\{x_1^{(1)},\dots,x_n^{(1)}\right\} \subset I$ and $\left\{\eta_1^{(1)},\dots,\eta_n^{(1)}\right\} \subset (0,1)$ be such that $\sum_{i=1}^{n}\eta_i^{(1)} = 1$. Then, for every $N \in \mathbb{N}$,*

$$\mathscr{J}_{(\phi,h)}(f,u^{(1)},\eta^{(1)}) \geq \sum_{k=1}^{N} h\left(n\eta_{\min}^{(k)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f\left(u_i^{(k)}\right) - f\left[\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(k)})\right)\right]\right), \tag{2.5}$$

where $\eta_i^{(k)}$ and $u_i^{(k)}$ are as in (2.3) and (2.4).

Proof. We demonstrate this by induction on N. For $N=1$, the result follows directly from Theorem 2.1. Suppose now that (2.5) holds for some $N \in \mathbb{N}$. Note that this implies that, for any convex sequence, $\left\{\eta_i^{(1)} : 1 \leq i \leq n\right\}$ and any elements $\left\{v_i^{(1)} : 1 \leq i \leq n\right\} \subset I$,

$$\begin{aligned} & f\left[\phi^{-1}\left(\sum_{i=1}^{n}\eta_i^{(1)}\phi(v_i^{(1)})\right)\right] + \sum_{k=1}^{N} h\left(n\eta_{\min}^{(k)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f\left(v_i^{(k)}\right) - f\left[\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(v_i^{(k)})\right)\right]\right) \\ & \leq \sum_{i=1}^{n} h\left(\theta_i^{(1)}\right) f\left(v_i^{(1)}\right). \end{aligned} \tag{2.6}$$

Then

$$\begin{aligned} A :=& \sum_{i=1}^{n} h\left(\eta_i^{(1)}\right) f\left(u_i^{(1)}\right) - h\left(n\eta_{\min}^{(1)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f\left(u_i^{(1)}\right) - f\left[\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(1)})\right)\right]\right) \\ &= \sum_{i=1}^{n} h\left(\eta_i^{(1)}\right) f\left(u_i^{(1)}\right) - h\left(n\eta_{\min}^{(1)}\right) h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f\left(u_i^{(1)}\right) \\ &+ h\left(\frac{|J_1|}{|J_1|}n\eta_{\min}^{(1)}\right) f\left[\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(1)})\right)\right] \\ &\geq \sum_{\substack{i=1 \\ \eta_i^{(1)} \neq \eta_{\min}^{(1)}}}^{n} h\left(\eta_i^{(1)} - \eta_{\min}^{(1)}\right) f\left(u_i^{(1)}\right) + |J_1|\left(h\left(\frac{1}{|J_1|}n\eta_{\min}^{(1)}\right) f\left[\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(1)})\right)\right]\right) \\ &= \sum_{\substack{i=1 \\ \eta_i^{(1)} \neq \eta_{\min}^{(1)}}}^{n} h\left(\eta_i^{(1)} - \eta_{\min}^{(1)}\right) f\left(u_i^{(1)}\right) + \sum_{\eta_i^{(1)} = \eta_{\min}^{(1)}}\left(h\left(\frac{1}{|J_1|}n\eta_{\min}^{(1)}\right) f\left[\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(1)})\right)\right]\right) \\ &= \sum_{i=1}^{n} h\left(\eta_i^{(2)}\right) f\left(u_i^{(2)}\right), \end{aligned} \tag{2.7}$$

where the last line is derived from the definitions of $\left(\eta_i^{(k)}\right)$ and $\left(u_i^{(k)}\right)$ in (2.3) and (2.4). For convenience, we denote $\eta_i^{(2)}$ by $\eta_i^{(1)}$ and $u_i^{(2)}$ by $v_i^{(1)}$. Note that

$$\begin{aligned}
\sum_{i=1}^{n}\eta_i^{(1)}=\sum_{i=1}^{n}\eta_i^{(2)} &= \sum_{i\notin J_1}\left(\eta_i^{(1)}-\eta_{\min}^{(1)}\right)+\sum_{i\in|J_1|}\frac{n\eta_{\min}^{(1)}}{|J_1|}\\
&=\sum_{i=1}^{n}\eta_i^{(1)}-\sum_{i\in J_1}\eta_i^{(1)}-\sum_{i\notin J_1}\eta_{\min}^{(1)}+n\eta_{\min}^{(1)}\\
&=1-|J_1|\,\eta_{\min}^{(1)}-(n-|J_1|)\,\eta_{\min}^{(1)}+n\eta_{\min}^{(1)}\\
&=1.
\end{aligned}$$

Therefore, we can use the inductive step (2.7) on (2.6) to derive

$$\begin{aligned}
I&=\sum_{i=1}^{n}h\left(\eta_i^{(1)}\right)f\left(v_i^{(1)}\right)\\
&\geq f\left(\phi^{-1}\left[\sum_{i=1}^{n}\eta_i^{(1)}\phi(v_i^{(1)})\right]\right)\\
&\quad+\sum_{k=1}^{N}h\left(n\eta_{\min}^{(k)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n}f\left(v_i^{(k)}\right)-f\left[\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(v_i^{(k)})\right)\right]\right).
\end{aligned}\tag{2.8}$$

Note that

$$\begin{aligned}
\phi^{-1}\left(\sum_{i=1}^{n}\eta_i^{(1)}\phi(v_i^{(1)})\right) &= \phi^{-1}\left(\sum_{i=1}^{n}\eta_i^{(2)}\phi(u_i^{(2)})\right)\\
&= \phi^{-1}\left(\sum_{\substack{i=1\\ i\notin J_1}}^{n}\left(\eta_i^{(1)}-\eta_{\min}^{(1)}\right)\phi(u_i^{(1)})+\sum_{j\in J_1}\left(\frac{n\eta_{\min}^{(1)}}{|J_1|}\sum_{i=1}^{n}\frac{\phi(u_i^{(1)})}{n}\right)\right)\\
&= \phi^{-1}\left(\sum_{\substack{i=1\\ i\notin J_1}}^{n}\eta_i^{(1)}\phi(u_i^{(1)})-\sum_{\substack{i=1\\ i\notin J_1}}^{n}\eta_{\min}^{(1)}\phi(u_i^{(1)})+\sum_{i=1}^{n}\eta_{\min}^{(1)}\phi(u_i^{(1)})\right)\\
&= \phi^{-1}\left(\sum_{\substack{i=1\\ i\notin J_1}}^{n}\eta_i^{(1)}\phi(u_i^{(1)})+\sum_{\substack{i=1\\ i\in J_1}}^{n}\eta_{\min}^{(1)}\phi(u_i^{(1)})\right)\\
&= \phi^{-1}\left(\sum_{i=1}^{n}\eta_i^{(1)}\phi(u_i^{(1)})\right).
\end{aligned}\tag{2.9}$$

Since $\eta_i^{(1)}=\eta_i^{(2)}$ and $v_i^{(1)}=u_i^{(2)}$, we have $\eta_i^{(k)}=\eta_i^{(k+1)}$ and $v_i^{(k)}=u_i^{(k+1)}$ for $k\geq 1$. Therefore, invoking (2.9) into (2.8), we have

$$\begin{aligned}
A&=\sum_{i=1}^{n}h\left(\eta_i^{(1)}\right)f\left(u_i^{(1)}\right)-h\left(n\eta_{\min}^{(1)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n}f\left(u_i^{(1)}\right)-f\left(\phi^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(1)})\right]\right)\right)\\
&\geq f\left[\phi^{-1}\left(\sum_{i=1}^{n}\eta_i^{(1)}(\phi(u_i))^{(1)}\right)\right]\\
&\quad+\sum_{k=1}^{N}h\left(n\eta_{\min}^{(k+1)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n}f\left(u_i^{(k+1)}\right)-f\left(\phi^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(k+1)})\right]\right)\right)\\
&=f\left[\phi^{-1}\left(\sum_{i=1}^{n}\eta_i^{(1)}\phi(u_i^{(1)})\right)\right]\\
&\quad+\sum_{k=2}^{N+1}h\left(n\eta_{\min}^{(k)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n}f\left(u_i^{(k)}\right)-f\left(\phi^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(k)})\right]\right)\right).
\end{aligned}$$

Thus,

$$\mathscr{J}_{(\phi,h)}(f,u^{(1)},\eta^{(1)})\geq\sum_{k=1}^{N+1}h\left(n\eta_{\min}^{(k)}\right)\left(h\left(\frac{1}{n}\right)\sum_{i=1}^{n}f\left(u_i^{(k)}\right)-f\left(\phi^{-1}\left[\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(k)})\right]\right)\right),$$

which completes the proof. □

Remark 2.2. Similar to [5, Remark 2.1 & Theorem 2.2], inequality (2.5) can be refined as

$$\mathscr{J}_{(\phi,h)}(f,u^{(1)},\eta^{(1)})\geq n\sum_{k=1}^{N}h\left(\eta_{\min}^{(k)}\right)\left(\frac{1}{n}\sum_{i=1}^{n}f\left(u_i^{(k)}\right)-f\left[\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\phi(u_i^{(k)})\right)\right]\right),$$

where the parameters are as in (2.3) and (2.4).

By substituting f with $\log f$ in Theorem 2.4, we derive the following multiplicative refinement for (ϕ,h) log-convex functions.

Corollary 2.4. *Let $f:I\to\mathbb{R}^+$ be a (ϕ,h)-log-convex function, and let $\left\{u_1^{(1)},\ldots,u_n^{(1)}\right\}\subset I$ and $\left\{\eta_1^{(1)},\ldots,\ \eta_n^{(1)}\right\}\subset(0,1)$ be such that $\sum_{i=1}^{n}\eta_i^{(1)}=1$. Then, for every $N\in\mathbb{N}$,*

$$\prod_{k=1}^{N}\left(\frac{\prod_{i=1}^{n}f^{h\left(\frac{1}{n}\right)}\left(u_i^{(k)}\right)}{f\left[\phi^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}(\phi(u_i))^{(k)}\right)\right]}\right)^{h\left(n\eta_{\min}^{(k)}\right)}\leq\frac{\prod_{i=1}^{n}f_i^{h\left(\eta_i^{(1)}\right)}\left(u_i^{(1)}\right)}{f\left[\phi^{-1}\left(\sum_{i=1}^{n}\eta_i^{(1)}(\phi(u_i))^{(1)}\right)\right]}.$$

3. FURTHER INEQUALITIES FOR (ϕ,h)-CONVEX FUNCTIONS

The aim of this section is to extend Theorems 2.1, 2.2, and 2.3 to a more general framework by using the so-called weak sub-majorization theory. Throughout this section, we denote by $U^*=\left(U_1^*,\ldots,U_n^*\right)$ the vector obtained from the vector $U=(U_1,\ldots,U_n)$ in $\mathbb{R}^n$ by rearranging the components of it in decreasing order. Then, for two vectors $U=(U_1,\ldots,U_n)$ and $V=(V_1,\ldots,V_n)$ in $\mathbb{R}^n$, V is said to be weakly sub-majorized by U, written $U\succ_w V$, if $\sum_{i=1}^k U_i^*\geq\sum_{i=1}^k V_i^*$ for all $k=1,\ldots,n$.

An important tool in the theory of weak sub-majorization, which will be used to prove our results, is provided by the following lemma.

Lemma 3.1. [10, pp. 13] *Let $U=(U_i)_{i=1}^n, V=(V_i)_{i=1}^n\in\mathbb{R}^n$ and $J\subset\mathbb{R}$ be an interval containing the components of U and V. If $U\succ_w V$ and $\psi:J\to\mathbb{R}$ is a continuous increasing convex function, then $\sum_{i=1}^n\psi(U_i)\geq\sum_{i=1}^n\psi(V_i)$.*

The following lemma allows us to derive the general form of Theorem 2.3.

Lemma 3.2. *Let f be a (ϕ,h)-convex function on $[a,b]$. Let $\{\eta_1,\ldots,\eta_n,\theta_1,\ldots,\theta_n\}\subset[0,1]$ such that $\sum_{i=1}^n\eta_i=\sum_{i=1}^n\theta_i=1$, and let $\{u_1,\ldots,u_n\}\subset(a,b)$. Define $U=(U_1,U_2,U_3)$ and $V=(V_1,V_2,V_3)$ as two vectors in $\mathbb{R}^3$ with components*

$$U_1=\sum_{i=1}^n h(\eta_i)f(u_i),\quad U_2=h(m)f\left(\phi^{-1}\left[\sum_{i=1}^n\theta_i\phi(u_i)\right]\right)$$

$$U_3=h(\mathbf{m}(|J|+1))f\left(\frac{1}{|J|+1}\phi^{-1}\left(\sum_{i\in J}\phi(u_i)+\phi\left(\sum_{i=1}^n\theta_iu_i\right)\right)\right)$$

$$V_1=f\left(\phi^{-1}\left[\sum_{i=1}^n\eta_i\phi(u_i)\right]\right),\quad \textit{and}\quad V_2=h(m)\sum_{i=1}^n h(\theta_i)f(u_i)$$

$$V_3=h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)\left[\sum_{i\in J}f(u_i)+f\left(\phi^{-1}\left(\sum_{i=1}^n\theta_i\phi(u_i)\right)\right)\right].$$

Then, $U\succ_w V$, namely, U^* and V^* have components satisfying

$$U_1^*\geq V_1^*, \tag{3.1}$$

$$U_1^*+U_2^*\geq V_1^*+V_2^*, \tag{3.2}$$

and

$$U_1^*+U_2^*+U_3^*\geq V_1^*+V_2^*+V_3^*. \tag{3.3}$$

Proof. Inequality (3.3) is a direct consequence of the first inequality in Theorem 2.3. Next, we prove inequality (3.1). It is evident from the definition of a (ϕ,h)-convex function that $U_1\geq V_1$, $V_2\geq U_2$, and $V_3\geq U_3$. Similarly, considering the non-negativity of f and defining m

as $\min_{1\leq j\leq n}\frac{\eta_j}{\theta_j}$, we can conclude that

$$\begin{aligned}U_1-V_2&=\sum_{i=1}^{n}h(\eta_i)f(u_i)-h(m)\sum_{i=1}^{n}h(\theta_i)f(u_i)\\&\geq\sum_{i=1}^{n}h(\eta_i-m\theta_i)f(u_i)\\&\geq 0,\end{aligned}$$

which yields $U_1\geq V_2$. Observe that

$$\begin{aligned}&\sum_{i=1}^{n}h(\eta_i)f(u_i)-h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)\sum_{i\in J}f(u_i)\\&\quad=\sum_{i\in J}h(\eta_i)f(u_i)-h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)\sum_{i\in J}f(u_i)+\sum_{i\notin J}h(\eta_i)f(u_i)\\&\quad\geq\sum_{i\in J}\left(h(\eta_i)-h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)\right)f(u_i)+\sum_{i\notin J}h(\eta_i)f(u_i)\\&\quad\geq\sum_{i\in J}h\left(\eta_i-\mathbf{m}\right)f(u_i)+\sum_{i\notin J}h(\eta_i)f(u_i)\\&\quad\geq\sum_{i\in J}h\left(m\theta_i\right)f(u_i)+\sum_{i\notin J}h(\eta_i)f(u_i)\\&\quad\geq\sum_{i=1}^{n}h\left(m\theta_i\right)f(u_i)\\&\quad\geq h(m)\sum_{i=1}^{n}h\left(\theta_i\right)f(u_i)\\&\quad\geq h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)\left[f\left(\phi^{-1}\left(\sum_{i=1}^{n}\theta_i\phi(u_i)\right)\right)\right],\end{aligned}$$

which indicates that $U_1\geq V_3$. This remains to show inequality (3.2). From the first inequality in Theorem 2.3, we have

$$U_1+U_2+U_3\geq V_1+V_2+V_3. \tag{3.4}$$

By using this and $V_3\geq U_3$, we have

$$V_1+V_2\leq U_1+U_2+U_3-V_3\leq U_1+U_2.$$

Furthermore, from inequality (3.4) and $V_2\geq U_2$, we deduce that

$$V_1+V_3\leq U_1+U_2+U_3-V_2\leq U_1+U_3.$$

Finally, we have

$$
\begin{aligned}
U_1+U_2-V_2 &= \sum_{i=1}^{n} h(\eta_i) f(u_i) + h(m) f\left(\phi^{-1}\left[\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right) - h(m)\sum_{i=1}^{n} h(\theta_i) f(u_i) \\
&= \sum_{i=1}^{n}\big(h(\eta_i)-h(m)h(\theta_i)\big) f(u_i) + h(m) f\left(\phi^{-1}\left[\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right) \\
&\geq \sum_{i=1}^{n} h(\eta_i - m\theta_i) f(u_i) + h(m) f\left(\phi^{-1}\left[\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right) \\
&= \sum_{i\in J} h(\eta_i - m\theta_i) f(u_i) + h(m) f\left(\phi^{-1}\left[\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right) \\
&\geq h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)\left[\sum_{i\in J} f(u_i) + f\left(\phi^{-1}\left(\sum_{i=1}^{n}\theta_i\phi(u_i)\right)\right)\right] = V_3.
\end{aligned}
$$

Thus $U_1+U_2 \geq V_2+V_3$. This yields $U \succ_w V$. $\square$

As an application of Lemma 3.2 and the theory of weak sub-majorization, we obtain the following extension of Theorem 2.3 to a more general form.

Theorem 3.1. *Let f be a positive (ϕ,h)-convex function on $[a,b]$, and ψ be a strictly increasing convex function defined on an interval J. Consider $u_1,\ldots,u_n \subset [a,b]$, $\{\eta_1,\ldots,\eta_n,\theta_1,\ldots,\theta_n\} \subset (0,1)$ such that $\sum_{i=1}^{n}\eta_i = \sum_{i=1}^{n}\theta_i = 1$. Then,*

$$
\begin{aligned}
&\psi\left(\sum_{i=1}^{n} h(\eta_i) f(u_i)\right) - \psi\circ f\left(\phi^{-1}\left[\sum_{i=1}^{n}\eta_i\phi(u_i)\right]\right) \\
&\geq \psi\left(h(m)\sum_{i=1}^{n} h(\theta_i) f(u_i)\right) - \psi\left(h(m) f\left(\phi^{-1}\left[\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right)\right) \\
&+ \psi\left(h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)\left[\sum_{i\in J} f(u_i) + f\left(\phi^{-1}\left(\sum_{i=1}^{n}\theta_i\phi(u_i)\right)\right)\right]\right) \\
&- \psi\left(h(\mathbf{m}(|J|+1)) f\left(\frac{1}{|J|+1}\phi^{-1}\left(\sum_{i\in J}\phi(u_i) + \phi\left(\sum_{i=1}^{n}\theta_i u_i\right)\right)\right)\right).
\end{aligned}
$$

Proof. Let us consider the vectors $U=(U_1,U_2,U_3)$ and $V=(V_1,V_2,V_3)$ defined in Lemma 3.2. We also have $U \succ_w V$, which implies by Lemma 3.1 that $\psi(U_1)+\psi(V_2)+\psi(V_3) \geq \psi(V_1)+\psi(V_2)+\psi(V_3)$, that is, $\psi(U_1)-\psi(V_1) \geq \psi(V_2)-\psi(U_2)+\psi(V_3)-\psi(U_3)$. $\square$

To establish a general form of Theorem 2.2, we first introduce the following useful lemma.

Lemma 3.3. *Let f be a positive (ϕ,h)-convex function on $[a,b]$, $\{\eta_1,\dots,\eta_n,\theta_1,\dots,\theta_n\}\subset[0,1]$ such that $\sum_{i=1}^n\eta_i=\sum_{i=1}^n\theta_i=1$ and $\{u_1,\dots,u_n\}\subset(a,b)$. Let $U=(U_1,U_2)$ and $V=(V_1,V_2)$ be two vectors in $\mathbb{R}^2$ with components*

$$U_1=\frac{1}{h\left(\frac{1}{M}\right)}\sum_{i=1}^n h(\theta_i)f(u_i),\;\; U_2=f\left(\phi^{-1}\left[\sum_{i=1}^n\eta_i\phi(u_i)\right]\right),$$

$$V_1=\sum_{i=1}^n h(\eta_i)f(u_i)\text{ and }V_2=\frac{1}{h\left(\frac{1}{M}\right)}f\left(\phi^{-1}\left[\sum_{i=1}^n\theta_i\phi(u_i)\right]\right).$$

Then, $U\succ_w V$, namely, U^* and V^* have components satisfying

$$U_1^*\geq V_1^* \tag{3.5}$$

and

$$U_1^*+U_2^*\geq V_1^*+V_2^*. \tag{3.6}$$

Proof. To prove (3.5), we note that $U_1^*=U_1$. Indeed, by using the super-multiplicativity of the function h, we see that

$$\begin{aligned}U_1-V_1&=\frac{1}{h\left(\frac{1}{M}\right)}\left(\sum_{i=1}^n h(\theta_i)f(u_i)-h\left(\frac{1}{M}\right)\sum_{i=1}^n h(\eta_i)f(u_i)\right)\\&\geq\frac{1}{h\left(\frac{1}{M}\right)}\sum_{i=1}^n\left(h\left(\theta_i-\frac{\eta_i}{M}\right)f(u_i)\right)\\&=\frac{1}{h\left(\frac{1}{M}\right)}\sum_{i=1}^n\left(h\left(\frac{M\theta_i-\eta_i}{M}\right)f(u_i)\right).\end{aligned}$$

At this point, we remark that $M\theta_i-\eta_i\geq\frac{\eta_i}{\theta_i}\theta_i-\eta_i=0$. For all $i=1,\dots,n$, we have $U_1\geq V_1$. Additionally, by the (ϕ,h)-convexity, it follows that $V_1\geq U_2$, which implies $U_1\geq U_2$ and consequently $U_1^*=U_1$. Furthermore, applying (ϕ,h)-convexity again, we can deduce that $U_1\geq V_2$. Inequality (3.6) directly follows from Theorem 2.2. □

Theorem 3.2. *Let f be a positive (ϕ,h)-convex function on $[a,b]$ and ψ be a strictly increasing convex function defined on an interval J. Let $\{u_1,\dots,u_n\}\subset[a,b]$, and $\{\eta_1,\dots,\eta_n,\theta_1,\dots,\theta_n\}\subset(0,1)$ be such that $\sum_{i=1}^n\eta_i=\sum_{i=1}^n\theta_i=1$. Then*

$$\begin{aligned}\psi\left(\sum_{i=1}^n h(\eta_i)f(u_i)\right)&\leq\psi\circ f\left(\phi^{-1}\left[\sum_{i=1}^n\eta_i\phi(u_i)\right]\right)\\&\quad+\psi\left(\frac{1}{h\left(\frac{1}{M}\right)}\sum_{i=1}^n h(\theta_i)f(u_i)\right)-\psi\left(\frac{1}{h\left(\frac{1}{M}\right)}f\left(\phi^{-1}\left[\sum_{i=1}^n\theta_i\phi(u_i)\right]\right)\right).\end{aligned}$$

Proof. Let us consider the vectors $U = (U_1, U_2)$ and $V = (V_1, V_2)$ defined in Lemma 3.3, which also yields $U \succ_w V$. This implies by Lemma 3.1 that $\psi(U_1) + \psi(U_2) \geq \psi(V_1) + \psi(V_2)$, that is,

$$\psi(V_1) \leq \psi(U_1) + \psi(U_2) - \psi(V_2).$$

$\square$

By replacing f with $\log f$, in Theorems 3.1 and 3.2, we can formulate the log-convex versions of the previous results as follows.

Theorem 3.3. *Let f be a positive (ϕ, h)-log-convex function on $[a,b]$, and let ψ be a strictly increasing convex function defined on an interval $[0,+\infty)$. Let $\{u_1,\ldots,u_n\} \subset [a,b]$ and $\{\eta_1,\ldots,\eta_n, \theta_1,\ldots,\theta_n\} \subset (0,1)$ be such that $\sum_{i=1}^n \eta_i = \sum_{i=1}^n \theta_i = 1$. Then*

$$\begin{aligned}
&\psi \circ \log\left(\prod_{i=1}^n f^{h(\eta_i)}(u_i)\right) - \psi \circ \log f\left(\phi^{-1}\left[\sum_{i=1}^n \eta_i \phi(u_i)\right]\right) \\
&\geq \psi\left(\log\left(\prod_{i=1}^n f^{h(\theta_i)}(u_i)\right)^{h(m)}\right) - \psi\left(\log f\left(\phi^{-1}\left[\sum_{i=1}^n \theta_i \phi(u_i)\right]\right)^{h(m)}\right) \\
&+ \psi\left(\log\left(\prod_{i=1}^n f(u_i)\right)^{h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)} + \log f\left(\phi^{-1}\left(\sum_{i=1}^n \theta_i \phi(u_i)\right)\right)^{h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)}\right) \\
&- \psi\left(\log f\left(\frac{1}{|J|+1}\phi^{-1}\left(\prod_{i=1}^n \phi(u_i) + \phi\left(\prod_{i=1}^n \theta_i u_i\right)\right)\right)^{h(\mathbf{m}(|J|+1))}\right),
\end{aligned}$$

and

$$\begin{aligned}
\psi \circ \log\left(\prod_{i=1}^n f^{h(\eta_i)}(u_i)\right) &\leq \psi \circ \log f\left(\phi^{-1}\left[\sum_{i=1}^n \eta_i \phi(u_i)\right]\right) \\
&+ \psi\left(\log\left(\prod_{i=1}^n f^{h(\theta_i)}(u_i)\right)^{\frac{1}{h\left(\frac{1}{M}\right)}}\right) - \psi\left(\log f\left(\phi^{-1}\left[\sum_{i=1}^n \theta_i \phi(u_i)\right]\right)^{\frac{1}{h\left(\frac{1}{M}\right)}}\right).
\end{aligned}$$

By choosing $\psi(x) = x^\lambda$ for $\lambda \geq 1$, we obtain the following more generalization of Theorems 3.1 and 3.2.

Theorem 3.4. *Let f be a positive (ϕ, h)-convex function on $[a,b]$, and let ψ be a strictly increasing convex function defined on an interval J. Let $\{u_1,\ldots,u_n\} \subset [a,b]$ and $\{\eta_1,\ldots,\eta_n, \theta_1,\ldots,\theta_n\}$*

$\subset (0,1)$ *be such that* $\sum_{i=1}^n \eta_i = \sum_{i=1}^n \theta_i = 1$. *Then*

$$\begin{aligned}
&\left(\sum_{i=1}^n h(\eta_i) f(u_i)\right)^\lambda - f^\lambda\left(\phi^{-1}\left[\sum_{i=1}^n \eta_i \phi(u_i)\right]\right)\\
&\geq \big(h(m)\big)^\lambda \left(\sum_{i=1}^n h(\theta_i) f(u_i)\right)^\lambda - \big(h(m)\big)^\lambda f^\lambda\left(\phi^{-1}\left[\sum_{i=1}^n \theta_i \phi(u_i)\right]\right)\\
&\quad + \left(h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)\right)^\lambda \left(\sum_{i\in J} f(u_i) + f\left(\phi^{-1}\left[\sum_{i=1}^n \theta_i\phi(u_i)\right]\right)\right)^\lambda\\
&\quad - \big(h(\mathbf{m}(|J|+1))\big)^\lambda f^\lambda\left(\frac{1}{|J|+1}\phi^{-1}\left(\sum_{i\in J}\phi(u_i) + \phi\left(\sum_{i=1}^n \theta_i u_i\right)\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\left(\sum_{i=1}^n h(\eta_i) f(u_i)\right)^\lambda \leq f^\lambda\left(\phi^{-1}\left[\sum_{i=1}^n \eta_i \phi(u_i)\right]\right)\\
&+ \left(\frac{1}{h\left(\frac{1}{M}\right)}\right)^\lambda \left(\sum_{i=1}^n h(\theta_i) f(u_i)\right)^\lambda - \left(\frac{1}{h\left(\frac{1}{M}\right)}\right)^\lambda f\left(\phi^{-1}\left[\sum_{i=1}^n \theta_i\phi(u_i)\right]\right)^\lambda.
\end{aligned}$$

By taking $\psi(x) = \exp(x)$, in Theorem 3.3, we obtain the following results, which present an additive refinement of the inequality (1.9).

Theorem 3.5. *Let f be a positive (ϕ,h)*-log-*convex function on $[a,b]$ and ψ be a strictly increasing convex function defined on an interval J. Let $\{u_1,\ldots,u_n\}\subset[a,b]$, $\{\eta_1,\ldots,\eta_n,\theta_1,\ldots,\theta_n\}\subset(0,1)$ be such that $\sum_{i=1}^n \eta_i = \sum_{i=1}^n \theta_i = 1$. Then*

$$\begin{aligned}
&\prod_{i=1}^n f^{h(\eta_i)}(u_i) - f\left(\phi^{-1}\left[\sum_{i=1}^n \eta_i\phi(u_i)\right]\right)\\
&\quad \geq \left(\prod_{i=1}^n f^{h(\theta_i)}(u_i)\right)^{h(m)} - f\left(\phi^{-1}\left[\sum_{i=1}^n \theta_i\phi(u_i)\right]\right)^{h(m)}\\
&\quad + \left(\prod_{i=1}^n f(u_i)\right)^{h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)} \times f\left(\phi^{-1}\left(\sum_{i=1}^n \theta_i\phi(u_i)\right)\right)^{h(\mathbf{m}(|J|+1))h\left(\frac{1}{|J|+1}\right)}\\
&\quad - f\left(\frac{1}{|J|+1}\phi^{-1}\left(\prod_{i=1}^n \phi(u_i) + \phi\left(\prod_{i=1}^n \theta_i u_i\right)\right)\right)^{h(\mathbf{m}(|J|+1))},
\end{aligned}$$

and

$$\prod_{i=1}^{n} f^{h(\eta_i)}(u_i) \le f\left(\phi^{-1}\left[\sum_{i=1}^{n}\eta_i\phi(u_i)\right]\right) + \left(\prod_{i=1}^{n} f^{h(\theta_i)}(u_i)\right)^{\frac{1}{h\left(\frac{1}{M}\right)}} - f\left(\phi^{-1}\left[\sum_{i=1}^{n}\theta_i\phi(u_i)\right]\right)^{\frac{1}{h\left(\frac{1}{M}\right)}}.$$

4. CONCLUSION

In this paper, we established several new refinements and extensions of Jensen's and Dragomir-type inequalities within the general framework of (φ, h)-convexity. Our results unify and improve a broad class of known inequalities, including those for convex, log-convex, h-convex, and (p,h)-convex functions, as special cases. Moreover, by employing weak sub-majorization techniques, we derived stronger bounds and additive–multiplicative refinements that enrich the theory of (φ, h)-convex functions and provide sharper estimates than those previously available in the literature. The results presented in this paper partially extend the results in [2, 6, 8, 11, 13, 14, 15, 16, 17, 18, 19, 21].

We believe that the inequalities and methods presented in this work may find further applications in functional analysis, operator theory, information theory, and related areas. Future research directions include extending these results to operator settings, integral forms, and other generalized convexity frameworks.

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